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In any Δ ABC the following relationship holds :

$$\sum_{cyc}\left(\frac{\sin\frac{A}{2}+\sin\frac{B}{2}}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}}\cdot\frac{\sin^2\frac{C}{2}\cdot\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}+\sin^2\frac{B}{2}\cdot\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{A}{2}+\sin\frac{B}{2}+2\sin\frac{C}{2}}\right)\geq 12$$

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$$\forall A',B',C',x',y',z'>0,$$

$$\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')\underset{①}{\overset{\text{Walter Janous}}{\geq}}\sqrt{3\sum_{cyc}A'B'}$$

$$\text{Now, }\sum_{cyc}\left(\frac{\sin\frac{A}{2}+\sin\frac{B}{2}}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}}\cdot\frac{\sin^2\frac{C}{2}\cdot\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}+\sin^2\frac{B}{2}\cdot\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{A}{2}+\sin\frac{B}{2}+2\sin\frac{C}{2}}\right)=$$

$$\sum_{cyc}\left(\frac{\sin\frac{A}{2}+\sin\frac{B}{2}}{\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)+\left(\sin\frac{C}{2}+\sin\frac{A}{2}\right)}\cdot\left(\frac{\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin^2\frac{B}{2}}+\frac{\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin^2\frac{C}{2}}\right)\right)$$

$$=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(\begin{array}{c}x'=\sin\frac{A}{2}+\sin\frac{B}{2},y'=\sin\frac{B}{2}+\sin\frac{C}{2},z'=\sin\frac{C}{2}+\sin\frac{A}{2},\\ A'=\frac{\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}}{\sin^2\frac{A}{2}},B'=\frac{\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin^2\frac{B}{2}},C'=\frac{\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin^2\frac{C}{2}}\end{array}\right)\overset{\text{via }①}{\geq}$$

$$\sqrt{3\sum_{cyc}\left(\frac{\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}}{\sin^2\frac{A}{2}}\cdot\frac{\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin^2\frac{B}{2}}\right)}\overset{\text{AM-GM}}{\geq}3.\sqrt[6]{\prod_{cyc}\left(\frac{\sin\frac{B}{2}+\sin\frac{C}{2}}{\sin^4\frac{A}{2}}\right)}\overset{\text{Cesaro}}{\geq}$$

$$3.\sqrt[6]{\frac{8}{\sin^3\frac{A}{2}\sin^3\frac{B}{2}\sin^3\frac{C}{2}}}=\frac{3\sqrt{2}}{\sqrt{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}}\overset{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\leq\frac{1}{8}}{\geq}\frac{3\sqrt{2}}{\sqrt{\frac{1}{8}}}=3\sqrt{2}.2\sqrt{2}=12$$

$$\text{and so, }\sum_{cyc}\left(\frac{\sin\frac{A}{2}+\sin\frac{B}{2}}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}}\cdot\frac{\sin^2\frac{C}{2}\cdot\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}+\sin^2\frac{B}{2}\cdot\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{A}{2}+\sin\frac{B}{2}+2\sin\frac{C}{2}}\right)\geq 12$$

$\forall\,\Delta$ ABC, $''=''$ iff Δ ABC is equilateral (QED)