ROMANIAN MATHEMATICAL MAGAZINE

In any acute triangle ABC the following relationship holds :

$$r_a\sqrt{\cos A}+r_b\sqrt{\mathrm{cst}\,B}+r_c\sqrt{\cos C}\le\frac{9\sqrt{2}}{4}R$$

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$$\sum_{cyc}\left(r_a.\sqrt{\cos A}\right)=s\sum_{cyc}\left(\sqrt{\tan\frac{A}{2}}.\sqrt{\tan\frac{A}{2}.\cos A}\right)\overset{CBS}{\le}$$

$$s.\sqrt{\sum_{cyc}\tan\frac{A}{2}}.\sqrt{\sum_{cyc}\left(\tan\frac{A}{2}.\left(2\cos^2\frac{A}{2}-1\right)\right)}=s.\sqrt{\frac{4R+r}{s}}.\sqrt{\sum_{cyc}\sin A-\sum_{cyc}\tan\frac{A}{2}}$$

$$=s.\sqrt{\frac{4R+r}{s}}.\sqrt{\frac{s}{R}-\frac{4R+r}{s}}=s.\sqrt{\frac{4R+r}{s}}.\sqrt{\frac{s^2-4R^2-Rr}{Rs}}\overset{Gerretsen}{\le}$$

$$\sqrt{\frac{4R+r}{R}}.\sqrt{3Rr+3r^2}=\sqrt{\frac{3r(4R+r)(R+r)}{R}}\overset{Euler}{\le}\sqrt{\frac{3.\frac{R}{2}.\frac{9R}{2}.\frac{3R}{2}}{R}}=9R.\sqrt{\frac{2}{16}}$$

$$=\frac{9\sqrt{2}}{4}R\ \text{and so},\ r_a.\sqrt{\cos A}+r_b.\sqrt{\mathrm{cst}\,B}+r_c.\sqrt{\cos C}\le\frac{9\sqrt{2}}{4}R\ \forall\ \text{acute}\ \Delta\ ABC,$$

$$''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$