ROMANIAN MATHEMATICAL MAGAZINE

In any acute triangle ABC the following relationship holds :

$$bc\sqrt{\cot A}+ca\sqrt{\cot B}+ab\sqrt{\cot C}>\frac{8R^2}{3}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Soumava Chakraborty-Kolkata-India

Let $\sqrt{\cot A}=x,\sqrt{\cot B}=y,\sqrt{\cot C}=z\ \therefore\ \sum_{cyc}\sqrt{\cot A}\overset{?}{>}2\Leftrightarrow\sum_{cyc}x>2$

$$\Leftrightarrow\left(\sum_{cyc}x\right)^2>4=4.\sqrt{\sum_{cyc}x^2y^2}\left(\because\sum_{cyc}\cot A\cot B=\sum_{cyc}x^2y^2=1\right)$$

$$\Leftrightarrow\left(\sum_{cyc}x\right)^4\underset{(*)}{\overset{?}{>}}16\sum_{cyc}x^2y^2$$

Assigning $y+z=X,z+x=Y,x+y=Z\Rightarrow X+Y-Z=2z>0,Y+Z-X=2x>0$ and $Z+X-Y=2y>0\Rightarrow X+Y>Z,Y+Z>X,Z+X>Y\Rightarrow X,Y,Z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say)

$$\Rightarrow 2\sum_{cyc}x=\sum_{cyc}X=2s\Rightarrow\sum_{cyc}x=s\to(i)\Rightarrow x=s-X,y=s-Y,z=s-Z\text{ and so}$$

$$\sum_{cyc}xy=\sum_{cyc}(s-X)(s-Y)\Rightarrow\sum_{cyc}xy=4Rr+r^2\to(ii)\text{ and }\sum_{cyc}x^2y^2=$$

$$\left(\sum_{cyc}xy\right)^2-2xyz\left(\sum_{cyc}x\right)\overset{\text{via (i) and (ii)}}{=}(4Rr+r^2)^2-2\left(\prod_{cyc}(s-X)\right).s$$

$$=(4Rr+r^2)^2-2r^2s.s\Rightarrow\sum_{cyc}x^2y^2=r^2((4R+r)^2-2s^2)\to(iii)\ \therefore\text{ via (i) \& (iii)},$$

$$(*)\Leftrightarrow s^4>16r^2((4R+r)^2-2s^2)\Leftrightarrow s^4+32r^2s^2\overset{(**)}{>}16r^2(4R+r)^2$$

Now, LHS of $(**)\overset{\text{Gerretsen}}{\geq}(16Rr+27r^2)s^2\overset{\text{Gerretsen}}{\geq}(16Rr+27r^2)(16Rr-5r^2)$

$\overset{?}{>}16r^2(4R+r)^2\Leftrightarrow 76r(R-2r)+148Rr+r^2\overset{?}{>}0\to$ true $\because R\overset{\text{Euler}}{\geq}2r$

$\Rightarrow(**)\Rightarrow(*)$ is true $\therefore\ \boxed{\sum_{cyc}\sqrt{\cot A}\overset{(\bullet)}{>}2}$ and WLOG we may assume $a\geq b\geq c$

and then : $bc\leq ca\leq ab$ and $\sqrt{\cot A}\leq\sqrt{\cot B}\leq\sqrt{\cot C}$

$$\therefore\ bc.\sqrt{\cot A}+ca.\sqrt{\cot B}+ab.\sqrt{\cot C}\overset{\text{Chebyshev}}{\geq}\frac{1}{3}\left(\sum_{cyc}bc\right)\left(\sum_{cyc}\sqrt{\cot A}\right)$$

$$\overset{\text{via }(\bullet)}{>}\frac{2(s^2+4Rr+r^2)}{3}>\frac{2(4R^2+4Rr+r^2)}{3}$$

$$\left(\begin{array}{c}\because \text{ABC is } a\text{cute} \Rightarrow \prod_{\text{cyc}} \cos A > 0 \Rightarrow \frac{s^2-(2R+r)^2}{4R^2} > 0 \\ \Rightarrow s > 2R + r > 2R \Rightarrow s^2 > 4R^2\end{array}\right) > \frac{8R^2}{3} \text{ } \textit{and so,}$$

$$bc.\sqrt{\cot A} + ca.\sqrt{\cot B} + ab.\sqrt{\cot C} > \frac{8R^2}{3} \ \forall \ a\text{cute } \Delta \ ABC \ (QED)$$