ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{R}{r}\sum_{cyc}\frac{1+\tan^2\frac{A}{2}\tan^2\frac{B}{2}}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^2}\ge 5$$

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$$\sum_{cyc}\frac{1+\tan^2\frac{B}{2}\tan^2\frac{C}{2}}{\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2}=\sum_{cyc}\frac{s^4+s^2(s-a)^2}{s^2\left(4R\cos^2\frac{A}{2}\right)^2}=\sum_{cyc}\frac{s^2+(s-a)^2}{16R^2.\frac{a^2s^2(s-a)^2}{16R^2r^2s^2}}$$

$$=r^2\sum_{cyc}\frac{s^2+(s-a)^2}{a^2(s-a)^2}=r^2\left(\sum_{cyc}\frac{(s-a)^2+a^2+2a(s-a)}{a^2(s-a)^2}+\sum_{cyc}\frac{1}{a^2}\right)$$

$$=r^2\left(\frac{2}{16R^2r^2s^2}\sum_{cyc}a^2b^2+\frac{1}{r^2s^2}\sum_{cyc}r_a^2+\frac{2}{4Rr}\sum_{cyc}\frac{bc}{s(s-a)}\right)$$

$$=r^2\left(\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{8R^2r^2s^2}+\frac{(4R+r)^2-2s^2}{r^2s^2}+\frac{2}{4Rr}.\frac{s^2+(4R+r)^2}{s^2}\right)$$

$$\left(\because\sum_{cyc}\frac{bc}{s(s-a)}=\sum_{cyc}\sec^2\frac{A}{2}\right)\therefore\frac{R}{r}\sum_{cyc}\frac{1+\tan^2\frac{A}{2}\tan^2\frac{B}{2}}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^2}$$

$$=\frac{(s^2+4Rr+r^2)^2-16Rrs^2+8R^2((4R+r)^2-2s^2)+4Rr(s^2+(4R+r)^2)}{8Rrs^2}\overset{?}{\ge}5$$

$$\Leftrightarrow s^4-(16R^2+44Rr-2r^2)s^2+128R^4+128R^3r+56R^2r^2+12Rr^3+r^4\ \boxed{\underset{(*)}{\overset{?}{\ge}}}\ 0$$

$$\text{Now, since } P=s^2(s^2-16Rr+5r^2)\overset{\text{Gerretsen}}{\ge}0\ \therefore\text{ in order to prove }(*),$$

$$\text{it suffices to prove : LHS of }(*)\overset{?}{\ge}P$$

$$\Leftrightarrow 128R^4+128R^3r+56R^2r^2+12Rr^3+r^4\ \boxed{\underset{(**)}{\overset{?}{\ge}}}\ (16R^2+28Rr+3r^2)s^2$$

$$\text{Finally, }(16R^2+28Rr+3r^2)s^2\overset{\text{Gerretsen}}{\le}(16R^2+28Rr+3r^2)(4R^2+4Rr+3r^2)$$

$$\overset{?}{\le}\text{LHS of }(**)\Leftrightarrow 16t^4-12t^3-29t^2-21t-2\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(16t^3+20t^2+11t+1)\overset{?}{\ge}0\to\text{true}\because t\overset{\text{Euler}}{\ge}2\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{R}{r}\sum_{cyc}\frac{1+\tan^2\frac{A}{2}\tan^2\frac{B}{2}}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^2}\ge 5\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$