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$I,\ I_a,\ I_b,\ I_c$ − *incenter and excenters of* ΔABC. *Prove that:*

$$\sum \frac{F_{I_aBC}}{AI \cdot AI_a} \ge \frac{3\sqrt{3}}{4}$$

Proposed by Sarkhan Adgozalov-Azerbaijan

Solution by Qurban Muellim-Azerbaijan

Lemma: $\sum \sin(2A) = 4\prod \sin A$

$$LHS = \sum \frac{F_{I_aBC}}{AI \cdot AI_a} = \sum \frac{a\sin^2\left(\frac{A}{2}\right)}{2r} = \frac{a+b+c-(a\cos A + b\cos B + c\cos C)}{4r} =$$

$$= \frac{2s - R\sum \sin(2A)}{4r} = \frac{2s - 4R\cdot\prod \sin A}{4r} = \frac{2s - 4R\cdot\frac{abc}{8R^3}}{4r} =$$

$$= \frac{2s - 4R\cdot\frac{4rsR}{8R^3}}{4r} = \frac{s}{2r} - \frac{s}{2R} \ge \frac{3\sqrt{3}r}{2r} - \frac{3\sqrt{3}R}{4R} = \frac{3\sqrt{3}}{4}$$

Equality holds for $a = b = c$.