ROMANIAN MATHEMATICAL MAGAZINE

Let Ω be the Brocard point of ΔABC.
O_a, O_b, O_c circumcenters of $\Delta B\Omega C$, $\Delta A\Omega C$, $\Delta A\Omega B$. Prove that:

$$\sum \frac{OO_a}{a \cdot AH} \geq \frac{2\sqrt{3}r}{R^2}$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Qurban Muellim-Azerbaijan

Lemma: $AH = 2R \cdot \cos A$

$$Let\ OO_{a \cap BC} = M,\ BM = MC.\ OM = x\ and\ O_aM = y.$$

$$\cot A = \frac{2x}{a},\ \cot C = \frac{2y}{a}.\ OO_a = x + y = \frac{a(\cot A + \cot C)}{2} =$$

$$= \frac{a\sin(A+C)}{2\sin A \sin C} = \frac{a \sin B}{2 \sin A \sin C} = \frac{a \cdot \frac{b}{2R}}{2 \cdot \frac{a}{2R} \cdot \frac{c}{2R}} = \frac{b}{c}R$$

$$LHS = \sum \frac{OO_a}{a \cdot AH} = \sum \frac{b}{2ac \cdot \cos A} = \sum \frac{b^2}{2abc \cos A} \geq \frac{\sum a^2}{2abc \sum \cos A} \geq$$

$$\geq \frac{4s^2}{2abc \cdot \frac{3}{2}} = \frac{4s^2}{3 \cdot 4Rrs} = \frac{s}{3Rr} \geq \frac{3\sqrt{3}r}{3R \cdot \frac{R}{2}} = \frac{2\sqrt{3}r}{R^2}$$

$$Equality\ holds\ for\ a = b = c.$$