ROMANIAN MATHEMATICAL MAGAZINE

Let $\Delta DEF$ be the orthic triangle of $\Delta ABC$.
$I,\ I_a,\ I_b,\ I_c$ – incenter and excenters of $\Delta ABC$. Prove that:

$$\sum \frac{EF}{AI\cdot AI_a}\le \frac{3R}{8\sqrt{3}r^2}$$

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$$\text{Lemmas: } EF=a\cdot\cos A,\ AI_a=\frac{r}{\sin\left(\frac{A}{2}\right)} \text{ and } AI=\frac{r}{\sin\left(\frac{A}{2}\right)}$$

$$\text{Schur İnequality: } a^n(a-b)(a-c)+b^n(b-a)(b-c)+c^n(c-a)(c-b)\ge 0$$

$$\text{For } n=3$$

$$a^5+b^5+c^5+abc(a^2+b^2+c^2)\ge a^4b+ab^4+b^4c+c^4b+c^4a+a^4c$$

$$a^4b+ab^4+b^4c+c^4b+c^4a+a^4c=ab(a^3+b^3)+bc(b^3+c^3)+ac(a^3+c^3)\ge$$
$$\ge a^2b^2(a+b)+b^2c^2(b+c)+a^2c^2(a+c)=$$
$$=a^3b^2+a^2b^3+b^3c^2+b^2c^3+c^3a^2+c^2a^3\Rightarrow$$

$$a^5+b^5+c^5+abc(a^2+b^2+c^2)\ge a^3b^2+a^2b^3+b^3c^2+c^3b^2+c^3a^2+a^3c^2$$

$$LHS=\sum\frac{EF}{AI\cdot AI_a}=\sum\frac{a\cos A}{\frac{rr_a}{\sin^2\left(\frac{A}{2}\right)}}=\sum\frac{a\cos A\cdot\sin^2\left(\frac{A}{2}\right)}{rr_a}=\sum\frac{a\cos A\cdot\sin^2\left(\frac{A}{2}\right)}{rs\tan\left(\frac{A}{2}\right)}=$$

$$=\sum\frac{a\cos A\sin A}{2rs}=\sum\frac{a^2\cos A}{4Rrs}=\frac{1}{4Rrs}\sum a^2\cos A=$$

$$=\frac{1}{4Rrs}\sum\frac{a^2b^2+a^2c^2-a^4}{2bc}=\frac{1}{4Rrs}\cdot\frac{\sum(a^3b^2+a^2b^3)-\sum a^5}{2abc}\le$$

$$\le\frac{1}{4Rrs}\cdot\frac{abc(a^2+b^2+c^2)}{2abc}=\frac{\sum a^2}{8Rrs}\le\frac{9R^2}{8Rrs}=\frac{9R}{8rs}\le\frac{9R}{24\sqrt{3}r^2}=\frac{3R}{8\sqrt{3}r^2}$$

Equality holds for $a=b=c$.