ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R^3}{r^3}\geq 8+\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

$$\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}\overset{CBS}{\leq}\sqrt{\left(\sum m_a^2\right)\left(\sum\frac{1}{m_a^2}\right)}\overset{m_a\geq\sqrt{s(s-a)}}{\leq}\sqrt{\left(\frac{3}{4}\sum a^2\right)\left(\sum\frac{1}{s(s-a)}\right)}\leq$$

$$\overset{Leibniz}{\leq}\sqrt{\left(\frac{3}{4}\times 9R^2\right)\left(\frac{4R+r}{s^2r}\right)}\overset{Doucet}{\leq}\sqrt{\left(\frac{3}{4}\times 9R^2\right)\left(\frac{4R+r}{3r(4R+r)r}\right)}=\frac{3R}{2r}$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\overset{AM-GM}{\geq}3$$

$$\textit{We need to show}:$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R^3}{r^3}.\geq 8+\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}$$

$$or, 3+\frac{R^3}{r^3}\geq 8+\frac{3R}{2r}\ or, 2x^3-3x-10\overset{\substack{\frac{R}{r}=x\geq 2\\ by\ Euler}}{\geq}0\ or, (x-2)(2x^2+4x+5)\geq 0\ true$$

Equality holds for an equilateral triangle.