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In ΔABC the following relationship holds:

$$\frac{a^4+b^4}{r_c}+\frac{b^4+c^4}{r_a}+\frac{c^4+a^4}{r_b}\geq 288r^3$$

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Solution by Daniel Sitaru – Romania

$$r_ar_br_c=\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}=\frac{F^3s}{s(s-a)(s-b)(s-c)}=\frac{F^3s}{F^2}=Fs$$

$$\frac{a^4+b^4}{r_c}+\frac{b^4+c^4}{r_a}+\frac{c^4+a^4}{r_b}=\sum_{cyc}\frac{a^4+b^4}{r_c}\overset{AM-GM}{\geq}$$

$$\geq 2\sum_{cyc}\frac{a^2b^2}{r_c}\overset{AM-GM}{\geq}2\cdot 3\cdot\sqrt[3]{\frac{(abc)^4}{r_ar_br_c}}=6abc\cdot\sqrt[3]{\frac{abc}{Fs}}=6\cdot 4RF\cdot\sqrt[3]{\frac{4RF}{Fs}}=$$

$$=24RF\sqrt[3]{\frac{4R}{s}}\overset{MITRINOVIC}{\geq}24RF\cdot\sqrt[3]{\frac{4}{s}\cdot\frac{2}{3\sqrt{3}}s}=$$

$$=24Rrs\cdot\frac{2}{\sqrt{3}}\overset{MITRINOVIC}{\geq}24R\cdot 3\sqrt{3}r\cdot\frac{2}{\sqrt{3}}=24\cdot 6Rr\overset{EULER}{\geq}144\cdot 2r\cdot r=288r^3$$

Equality holds for $a=b=c$.