ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{r_a^2+r_b^2}{c}+\frac{r_b^2+r_c^2}{a}+\frac{r_c^2+r_a^2}{b}\ge 9\sqrt{3}r$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{r_a^2+r_b^2}{c}+\frac{r_b^2+r_c^2}{a}+\frac{r_c^2+r_a^2}{b}=\sum_{cyc}\frac{r_a^2+r_b^2}{c}=\sum_{cyc}\frac{r_a^2}{c}+\sum_{cyc}\frac{r_b^2}{c}\overset{BERGSTROM}{\ge}$$

$$\ge\frac{(r_a+r_b+r_c)^2}{a+b+c}+\frac{(r_a+r_b+r_c)^2}{a+b+c}=\frac{2(r_a+r_b+r_c)^2}{a+b+c}=$$

$$=\frac{2(4R+r)^2}{2s}=\frac{(4R+r)^2}{s}\overset{DOUCET}{\ge}\frac{(4R+r)^2}{\frac{4R+r}{\sqrt{3}}}=\sqrt{3}(4R+r)\overset{EULER}{\ge}$$

$$\ge\sqrt{3}(4\cdot 2r+r)=9\sqrt{3}r$$

Equality holds for $a=b=c$.