ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{m_a^2+m_b^2}{c^2}+\frac{m_b^2+m_c^2}{a^2}+\frac{m_c^2+m_a^2}{b^2}\ge\frac{9}{2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{m_a^2+m_b^2}{c^2}+\frac{m_b^2+m_c^2}{a^2}+\frac{m_c^2+m_a^2}{b^2}=$$

$$=\sum_{cyc}\frac{m_a^2+m_b^2}{c^2}=\sum_{cyc}\frac{2(b^2+c^2)-a^2+2(a^2+c^2)-b^2}{4c^2}=$$

$$=\sum_{cyc}\frac{b^2+a^2+4c^2}{4c^2}=\frac{1}{4}\sum_{cyc}\frac{b^2}{c^2}+\frac{1}{4}\sum_{cyc}\frac{a^2}{c^2}+\sum_{cyc}\frac{4c^2}{4c^2}\ge$$

$$\overset{AM-GM}{\ge}\frac{1}{4}\cdot 3\sqrt[3]{\frac{b^2}{c^2}\cdot\frac{c^2}{a^2}\cdot\frac{a^2}{b^2}}+\frac{1}{4}\cdot 3\sqrt[3]{\frac{a^2}{c^2}\cdot\frac{b^2}{a^2}\cdot\frac{c^2}{b^2}}+3=$$

$$=\frac{3}{4}+\frac{3}{4}+3=\frac{6}{4}+3=\frac{3}{2}+3=\frac{9}{2}$$

Equality holds for $a=b=c$.