ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{m_a^2+m_b^2}{c}+\frac{m_b^2+m_c^2}{a}+\frac{m_c^2+m_a^2}{b}\ge 9\sqrt{3}r$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{m_a^2+m_b^2}{c}+\frac{m_b^2+m_c^2}{a}+\frac{m_c^2+m_a^2}{b}=$$

$$=\sum_{cyc}\frac{m_a^2+m_b^2}{c}=\sum_{cyc}\frac{2(b^2+c^2)-a^2+2(a^2+c^2)-b^2}{4c}=$$

$$=\sum_{cyc}\frac{b^2+a^2+4c^2}{4c}=\frac{1}{4}\sum_{cyc}\frac{b^2}{c}+\frac{1}{4}\sum_{cyc}\frac{a^2}{c}+\sum_{cyc}c\ge$$

$$\overset{BERGSTROM}{\ge}\frac{1}{4}\cdot\frac{(a+b+c)^2}{a+b+c}+\frac{1}{4}\cdot\frac{(a+b+c)^2}{a+b+c}+2s=$$

$$=\frac{a+b+c}{4}+\frac{a+b+c}{4}+2s=\frac{2s}{4}+\frac{2s}{4}+2s=$$

$$=\frac{4s}{4}+2s=3s\overset{MITRINOVIC}{\ge}3\cdot 3\sqrt{3}r=9\sqrt{3}r$$

Equality holds for $a=b=c$.