ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{r_a+r_b}{h_c}+\frac{r_b+r_c}{h_a}+\frac{r_c+r_a}{h_b}\le\frac{3R}{r}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sum_{cyc}\frac{r_a+r_b}{h_c}=\sum_{cyc}\frac{\frac{F}{s-a}+\frac{F}{s-b}}{\frac{2F}{c}}=\sum_{cyc}\frac{\frac{1}{s-a}+\frac{1}{s-b}}{\frac{2}{c}}=$$

$$=\frac{1}{2}\sum_{cyc}\frac{c(s-a+s-b)}{(s-a)(s-b)}=\frac{1}{2}\sum_{cyc}\frac{c(a+b+c-a-b)}{(s-a)(s-b)}=$$

$$=\frac{1}{2}\sum_{cyc}\frac{c^2}{(s-a)(s-b)}=\frac{1}{2(s-a)(s-b)(s-c)}\sum_{cyc}c^2(s-c)=$$

$$=\frac{s}{2s(s-a)(s-b)(s-c)}\cdot 4rs(R+r)=$$

$$=\frac{4rs^2(R+r)}{2F^2}=\frac{2rs^2(R+r)}{r^2s^2}=\frac{2(R+r)}{r}\le\frac{2\left(R+\frac{R}{2}\right)}{r}=\frac{2\cdot\frac{3R}{2}}{r}=\frac{3R}{r}$$

Equality holds for $a=b=c$.