ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{(a+b)^2}{r_c}+\frac{(b+c)^2}{r_a}+\frac{(c+a)^2}{r_b}\geq 48r$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{(a+b)^2}{r_c}+\frac{(b+c)^2}{r_a}+\frac{(c+a)^2}{r_b}=\sum_{cyc}\frac{(a+b)^2}{r_c}\geq$$

$$\overset{AM-GM}{\geq}3\sqrt[3]{\frac{\big((a+b)(b+c)(c+a)\big)^2}{r_ar_br_c}}\overset{CESARO}{\geq}3\sqrt[3]{\frac{(8abc)^2}{\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}}}=3\sqrt[3]{\frac{64(abc)^2}{\frac{F^3s}{F^2}}}=$$

$$=12\sqrt[3]{\frac{(4RF)^2}{Fs}}=12\sqrt[3]{\frac{16R^2F^2}{Fs}}=12\sqrt[3]{\frac{16R^2F}{s}}=12\sqrt[3]{\frac{16R^2rs}{s}}=$$

$$=12\sqrt[3]{16R^2r}\overset{EULER}{\geq}12\sqrt[3]{16\cdot 4r^2\cdot r}=12\sqrt[3]{64r^3}=12\cdot 4r=48r$$

Equality holds for $a=b=c$.