ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{(r_a+r_b)^2}{h_c}+\frac{(r_b+r_c)^2}{h_a}+\frac{(r_c+r_a)^2}{h_b}\geq 36r$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{(r_a+r_b)^2}{h_c}+\frac{(r_b+r_c)^2}{h_a}+\frac{(r_c+r_a)^2}{h_b}=\sum_{cyc}\frac{(r_a+r_b)^2}{h_c}\geq$$

$$\overset{AM-GM}{\geq} 3\sqrt[3]{\frac{\left((r_a+r_b)(r_b+r_c)(r_c+r_a)\right)^2}{h_ah_bh_c}}\overset{CESARO}{\geq} 3\sqrt[3]{\frac{(8r_ar_br_c)^2}{h_ah_bh_c}}=$$

$$=3\sqrt[3]{\frac{64\cdot\left(\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}\right)^2}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}}=12\sqrt[3]{\frac{F^6}{\left((s-a)(s-b)(s-c)\right)^2}\cdot\frac{abc}{8F^3}}=$$

$$=\frac{12}{2}\sqrt[3]{\frac{F^3\cdot 4RF\cdot s^2}{\left(s(s-a)(s-b)(s-c)\right)^2}}=6\sqrt[3]{\frac{4Rs^2F^4}{F^4}}=$$

$$=6\sqrt[3]{4Rs^2}\overset{MITRINOVIC}{\geq}6\sqrt[3]{4R\cdot 27r^2}\overset{EULER}{\geq}6\cdot 3\sqrt[3]{8r\cdot r^2}=36r$$

Equality holds for $a=b=c$.