ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc} \frac{a+b}{\sqrt{r_c}} \le 2\sum_{cyc} \frac{a}{\sqrt{h_a}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $\sqrt{s-a} = x, \sqrt{s-b} = y, \sqrt{s-c} = z \; (\because (s-a),(s-b),(s-c) > 0)$

$and\ then: s-a = x^2, s-b = y^2, s-c = z^2 \Rightarrow 3s - 2s = \sum_{cyc} x^2$ and so,

$$a = y^2+z^2, b = z^2+x^2, c = x^2+y^2 \therefore \sum_{cyc} \frac{b+c}{\sqrt{r_a}} \overset{?}{\le} 2\sum_{cyc} \frac{a}{\sqrt{h_a}} \Leftrightarrow$$

$$\sum_{cyc} \frac{x(2x^2+y^2+z^2)}{\sqrt{rs}} \overset{?}{\le} \sum_{cyc} \frac{(y^2+z^2).\sqrt{2(y^2+z^2)}}{\sqrt{rs}}$$

$$\Leftrightarrow \sum_{cyc}\left((y^2+z^2).\sqrt{2(y^2+z^2)}\right) \boxed{\overset{?}{\underset{(*)}{\geq}}} \sum_{cyc}\left(x(2x^2+y^2+z^2)\right)$$

$$\text{Now, } \sum_{cyc}\left((y^2+z^2).\sqrt{2(y^2+z^2)}\right) \ge$$

$$\sum_{cyc}\left((y^2+z^2)(y+z)\right) \overset{?}{=} \sum_{cyc}\left(x(2x^2+y^2+z^2)\right)$$

$$\Leftrightarrow \sum_{cyc}\left((y^2+z^2)\left(\sum_{cyc} x - x\right)\right) \overset{?}{=} \left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x\right) + \sum_{cyc} x^3$$

$$\Leftrightarrow 2\left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x\right) - P \overset{?}{=} 2\sum_{cyc} x^3 + P \quad \left(P = \sum_{cyc} x^2 y + \sum_{cyc} xy^2\right)$$

$$\Leftrightarrow 2\sum_{cyc} x^3 + 2P - P \overset{?}{=} 2\sum_{cyc} x^3 + P \rightarrow \text{true} \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{a+b}{\sqrt{r_c}} \le 2\sum_{cyc} \frac{a}{\sqrt{h_a}} \; \forall \Delta \text{ABC}, '' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$