ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{r_a+2r_b}{h_c}+\frac{r_b+2r_c}{h_a}+\frac{r_c+2r_a}{h_b}\ge 9$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{r_a+2r_b}{h_c}+\frac{r_b+2r_c}{h_a}+\frac{r_c+2r_a}{h_b}=\sum_{cyc}\frac{r_a+2r_b}{h_c}=\sum_{cyc}\frac{r_a}{h_c}+2\sum_{cyc}\frac{r_b}{h_c}\overset{AM-GM}{\ge}$$

$$\ge 3\sqrt[3]{\frac{r_ar_br_c}{h_ah_bh_c}}+2\cdot 3\sqrt[3]{\frac{r_ar_br_c}{h_ah_bh_c}}=9\sqrt[3]{\frac{r_ar_br_c}{h_ah_bh_c}}\ge 9\Leftrightarrow\sqrt[3]{\frac{r_ar_br_c}{h_ah_bh_c}}\ge 1\Leftrightarrow r_ar_br_c\ge h_ah_bh_c$$

$$\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}\ge\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}$$

$$abc\ge 8(s-a)(s-b)(s-c)$$

$$abc\cdot s\ge 8s(s-a)(s-b)(s-c)$$

$$4RF\cdot s\ge 8F^2,\qquad Rs\ge 2F,\qquad Rs\ge 2rs$$

$$R\ge 2r\quad \text{(Euler)}$$

Equality holds for $a=b=c$.