ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$(h_a+h_b)^{\cos\frac{C}{2}}\cdot(h_b+h_c)^{\cos\frac{A}{2}}\cdot(h_c+h_a)^{\cos\frac{B}{2}}\le(3R)^{\frac{3\sqrt{3}}{2}}$$

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$$\text{WLOG: } a\le b\le c\Rightarrow\frac{1}{a}\ge\frac{1}{b}\ge\frac{1}{c}\Rightarrow\frac{2F}{a}\ge\frac{2F}{b}\ge\frac{2F}{c}$$

$$\Rightarrow h_a\ge h_b\ge h_c\Rightarrow\begin{cases}h_a+h_c\ge h_b+h_c\\ h_b+h_a\ge h_c+h_a\end{cases}\Rightarrow h_a+h_b\ge h_a+h_c\ge h_b+h_c\quad(1)$$

$$a\le b\le c\Rightarrow\cos\frac{A}{2}\ge\cos\frac{B}{2}\cos\frac{C}{2}\Rightarrow\cos\frac{C}{2}\le\cos\frac{B}{2}\le\cos\frac{A}{2}\quad(2)$$

By (1); (2) and Cebyshev's inequality:

$$(h_a+h_b)\cos\frac{C}{2}+(h_b+h_c)\cos\frac{A}{2}+(h_c+h_a)\cos\frac{A}{2}\le$$

$$\le\frac{1}{3}(h_a+h_b+h_b+h_c+h_c+h_a)\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)=$$

$$=\frac{2}{3}(h_a+h_b+h_c)\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)\quad(3)$$

$$\prod_{cyc}(h_a+h_b)\cos\frac{C}{2}\overset{WEIGHTED\ AM-GM}{\le}$$

$$\le\left(\frac{(h_a+h_b)\cos\frac{C}{2}+(h_b+h_c)\cos\frac{A}{2}+(h_c+h_a)\cos\frac{B}{2}}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}\right)^{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}\le$$

$$\overset{(3)}{\le}\left(\frac{\frac{2}{3}(h_a+h_b+h_c)\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}\right)^{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}\le$$

$$\overset{Jensen}{\le}\left(\frac{2}{3}\left(\frac{2F}{a}+\frac{2F}{b}+\frac{2F}{c}\right)\right)^{3\cos\frac{A+B+C}{6}}=\left(\frac{4F}{3}\cdot\frac{ab+bc+ca}{abc}\right)^{3\cos\frac{\pi}{6}}=$$

$$=\left(\frac{4F}{3}\cdot\frac{s^2+r^2+4Rr}{4RF}\right)^{\frac{3\sqrt{3}}{2}}=\left(\frac{s^2+r^2+4Rr}{3R}\right)^{\frac{3\sqrt{3}}{2}}\le$$

$$\overset{MITRINOVIC}{\leq}\left(\frac{\frac{27R^2}{4}+r^2+4Rr}{3R}\right)^{\frac{3\sqrt{3}}{2}}\overset{EULER}{\leq}$$

$$\leq\left(\frac{\frac{27R^2}{4}+\frac{R^2}{4}+4R\cdot\frac{R}{2}}{3R}\right)^{\frac{3\sqrt{3}}{2}}=\left(\frac{7R^2+2R^2}{3R}\right)^{\frac{3\sqrt{3}}{2}}=(3R)^{\frac{3\sqrt{3}}{2}}$$

Equality holds for $a=b=c$.