ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$216r^3 \leq (r_a + r_b)(r_b + r_c)(r_c + r_a) \leq 27R^3$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$(r_a + r_b)(r_b + r_c)(r_c + r_a) = \prod_{cyc}(r_a + r_b) =$$

$$= \prod_{cyc}\left(\frac{F}{s-a} + \frac{F}{s-b}\right) = F^3\prod_{cyc}\left(\frac{1}{s-a} + \frac{1}{s-b}\right) =$$

$$= F^3\prod_{cyc}\frac{s-b+s-a}{(s-a)(s-b)} = F^3\prod_{cyc}\frac{2s-a-b}{(s-a)(s-b)} =$$

$$= F^3\prod_{cyc}\frac{a+b+c-a-b}{(s-a)(s-b)} = F^3\prod_{cyc}\frac{c}{(s-a)(s-b)} =$$

$$= F^3 \cdot \frac{abc}{(s-a)^2(s-b)^2(s-c)^2} = \frac{F^3 \cdot abc \cdot s^2}{s^2(s-a)^2(s-b)^2(s-c)^2} = \frac{F^3 \cdot 4RF \cdot s^2}{F^4} = 4Rs^2$$

Remains to prove:

$$216r^3 \leq 4Rs^2 \leq 27R^3$$

$$4Rs^2 \overset{MITRINOVIC}{\leq} 4R \cdot \left(\frac{3\sqrt{3}}{2}R\right)^2 = 4R \cdot \frac{27R^2}{4} = 27R^3$$

$$4Rs^2 \overset{MITRINOVIC}{\geq} 4R \cdot \left(3\sqrt{3}r\right)^2 = 4R \cdot 27r^2 \overset{EULER}{\geq} 4 \cdot 2r \cdot 27r^2 = 216r^3$$

Equality holds for $a = b = c$.