ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$216r^3 \leq (h_a + h_b)(h_b + h_c)(h_c + h_a) \leq 27R^3$$

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$$(h_a + h_b)(h_b + h_c)(h_c + h_a) = \prod_{cyc}(h_a + h_b) =$$

$$= \prod_{cyc}\left(\frac{2F}{a} + \frac{2F}{b}\right) = 8F^3 \prod_{cyc}\left(\frac{1}{a} + \frac{1}{b}\right) =$$

$$= 8F^3 \prod_{cyc}\left(\frac{a+b}{ab}\right) = 8F^3 \cdot \frac{(a+b)(b+c)(c+a)}{abc \cdot abc} \overset{CESARO}{\geq}$$

$$\geq 8F^3 \cdot \frac{8abc}{abc \cdot abc} = \frac{64F^3}{abc} = \frac{64F^3}{4RF} = \frac{16F^2}{R} = \frac{16r^2s^2}{R}$$

Remains to prove:

$$216r^3 \leq \frac{16r^2s^2}{R} \leq 27R^3$$

$$\frac{16r^2s^2}{R} \overset{EULER}{\leq} \frac{16 \cdot \left(\frac{R}{2}\right)^2 \cdot s^2}{R} = 4Rs^2 \overset{MITRINOVIC}{\leq} 4R \cdot \left(\frac{3\sqrt{3}}{2}R\right)^2 = 4R \cdot \frac{27R^2}{4} = 27R^3$$

$$\frac{16r^2s^2}{R} \geq 216r^3 \Leftrightarrow \frac{2r^2s^2}{R} \geq 27r^3 \Leftrightarrow \frac{2s^2}{R} \geq 27r \Leftrightarrow 2s^2 \geq 27Rr \quad \text{(to prove)}$$

$$2s^2 \overset{GERRESTEN}{\geq} 2(16Rr - 5r^2) = 32Rr - 10r^2 \geq 27Rr \Leftrightarrow$$

$$\Leftrightarrow 5Rr \geq 10r^2 \Leftrightarrow R \geq 2r \quad \text{(Euler)}$$

Equality holds for $a = b = c$.