ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds

$$\frac{a+b}{\sin^2 C}+\frac{b+c}{\sin^2 A}+\frac{c+a}{\sin^2 B}\ge 8\sqrt{3}R$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a+b}{\sin^2 C}+\frac{b+c}{\sin^2 A}+\frac{c+a}{\sin^2 B}=\sum_{cyc}\frac{a+b}{\sin^2 C}\ge 3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{\sin^2 A\sin^2 B\sin^2 C}}\overset{CESARO}{\ge}$$

$$\ge 3\sqrt[3]{\frac{8abc}{\frac{a^2}{4R^2}\cdot\frac{b^2}{4R^2}\cdot\frac{c^2}{4R^2}}}=3\cdot 4R^2\sqrt[3]{\frac{8abc}{a^2b^2c^2}}=12R^2\cdot 2\cdot\frac{1}{\sqrt[3]{abc}}=\frac{24R^2}{\sqrt[3]{4Rrs}}\ge$$

$$\overset{EULER}{\ge}\frac{24R^2}{\sqrt[3]{4R\cdot\frac{R}{2}\cdot s}}=\frac{24R^2}{\sqrt[3]{2R^2\cdot\frac{3\sqrt{3}}{2}R}}=\frac{24R^2}{\sqrt[3]{\left(\sqrt{3}R\right)^3}}=\frac{24R^2}{\sqrt{3}R}=\frac{24R}{\sqrt{3}}=\frac{24\sqrt{3}R}{3}=8\sqrt{3}R$$

Equality holds for $a=b=c$.