ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$(m_a+m_b)^c.(m_c+m_b)^a.(m_a+m_c)^b \ge (6r)^{6r\sqrt{3}}$$

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Solution by Mirsadix Muzefferov-Azerbaijan

$$(m_a+m_b)^c.(m_c+m_b)^a.(m_a+m_c)^b \overset{Weighted\ \ GM-HM}{\ge}$$

$$\ge\left(\frac{a+b+c}{\frac{a}{m_c+m_b}+\frac{b}{m_a+m_c}+\frac{c}{m_a+m_b}}\right)^{\sum a}\ge$$

$$In\ \Delta ABC\ wlog:\ a\le b\le c,\qquad m_c\le m_b\le m_a$$

$$\to\begin{cases}\frac{1}{m_a+m_b}\le\frac{1}{m_a+m_c}\le\frac{1}{m_c+m_b}\\ c\ge b\ge a\end{cases}$$

$$\overset{Chebyshev}{\ge}\left(\frac{a+b+c}{\frac{1}{3}(a+b+c)\left(\frac{1}{m_c+m_b}+\frac{1}{m_a+m_c}+\frac{1}{m_a+m_b}\right)}\right)^{\sum a}=$$

$$=\left(\frac{3}{\frac{1}{m_c+m_b}+\frac{1}{m_a+m_c}+\frac{1}{m_a+m_b}}\right)^{\sum a}\overset{\left\{\frac{1}{m_c+m_b}\le\frac{1}{4}\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\right\}}{\ge}$$

$$\left(\frac{3}{\frac{1}{4}.2\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)}\right)^{\sum a}=\left(\frac{6}{\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}}\right)^{\sum a}\ge\left(\frac{6}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}\right)^{\sum a}=$$

$$=(6r)^{\sum a}\overset{Mitrinovici}{\ge}(6r)^{6r\sqrt{3}}$$

$$Equality\ \ holds\ \ if\ \ a=b=c$$