ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$(r_a+r_b)^c.(r_b+r_c)^a.(r_a+r_c)^b \le (3R)^{3R\sqrt{3}}$$

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$$In\ \Delta ABC\ wlog:\ a\le b\le c,\qquad r_a\le r_b\le r_c$$

$$\to \begin{cases} r_a+r_b\le r_a+r_c\le r_b+r_c \\ c\ge b\ge a \end{cases}$$

$$(r_a+r_b)^c.(r_b+r_c)^a.(r_a+r_c)^b \overset{Weighted\ \ GM-HM}{\overbrace{\le}}$$

$$\le\left(\frac{c(r_a+r_b)+a(r_b+r_c)+b(r_a+r_c)}{a+b+c}\right)^{\sum a} \overset{Chebyshev}{\overbrace{\le}}$$

$$\le\left(\frac{\frac{1}{3}(a+b+c)\big(2(r_a+r_b+r_c)\big)}{a+b+c}\right)^{a+b+c} =$$

$$=\left(\frac{2}{3}(4R+r)\right)^{a+b+c} \overset{Euler}{\overbrace{\le}} \left(\frac{2}{3}\left(4R+\frac{R}{2}\right)\right)^{2p} = (3R)^{2p} \overset{Mitrinovici}{\overbrace{\le}} (3R)^{3R\sqrt{3}}$$

$$Equality\ \ holds\ \ if\ \ a=b=c.$$