ROMANIAN MATHEMATICAL MAGAZINE

In *any acute* Δ ABC the *following relationship holds* :

$$\sum_{cyc} \tan^2 A \ge \sum_{cyc} \cot^2 \frac{A}{2}$$

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$$\sum_{cyc} \tan^2 A = \left(\sum_{cyc} \tan A\right)^2 - 2\sum_{cyc} \tan A \tan B$$

$$= \left(\frac{\prod_{cyc} \sin A}{\prod_{cyc} \cos A}\right)^2 - 2\left(\frac{\prod_{cyc} \sin A}{\prod_{cyc} \cos A}\right)\left(\sum_{cyc} \cot A\right)$$

$$= \left(\frac{\frac{4Rrs}{8R^3}}{\frac{s^2-4R^2-4Rr-r^2}{4R^2}}\right)^2 - 2\left(\frac{\frac{4Rrs}{8R^3}}{\frac{s^2-4R^2-4Rr-r^2}{4R^2}}\right)\left(\frac{s^2-4Rr-r^2}{2rs}\right)$$

$$= \frac{4r^2s^2 - 2(s^2-4R^2-4Rr-r^2)(s^2-4Rr-r^2)}{(s^2-4R^2-4Rr-r^2)^2} \overset{?}{\ge} \sum_{cyc} \cot^2 \frac{A}{2}$$

$$= \sum_{cyc} \frac{s^2}{r_a^2} = s^2\left(\frac{1}{r^2} - \frac{2(4R+r)}{s^2r}\right) = \frac{s^2-8Rr-2r^2}{r^2}$$

$$\Leftrightarrow -s^6 + (8R^2+16Rr+2r^2)s^4 - (16R^4+96R^3r+96R^2r^2+24Rr^3-3r^4)s^2$$

$$+8Rr(16R^4+36R^3r+28R^2r^2+9Rr^3+r^4) \underset{(*)}{\overset{?}{\ge}} 0$$

$$\text{and} \because P = -(s^2-4R^2-4Rr-r^2)(s^2-2R^2-8Rr-3r^2)(s^2-4R^2-4Rr-3r^2)$$

$$\underset{\text{Gerretsen}}{\overset{\text{Walker and}}{\ge}} 0 \ (\text{and} \because s > 2R+r) \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$\text{LHS of } (*) \overset{?}{\ge} P \Leftrightarrow -(2R^2+5r^2)s^4 + (16R^4+16R^3r+32R^2r^2+48Rr^3+18r^4)s^2 -$$

$$(32R^6+64R^5r+80R^4r^2+160R^3r^3+158R^2r^4+64Rr^5+9r^6) \underset{(**)}{\overset{?}{\ge}} 0$$

$$\text{and} \because Q = -(2R^2+5r^2)(s^2-4R^2-4Rr-r^2)(s^2-4R^2-4Rr-3r^2) \underset{\text{and Gerretsen}}{\overset{s>2R+r}{\ge}} 0$$

$$\therefore \text{in order to prove } (**), \text{it suffices to prove : LHS of } (**) \overset{?}{\ge} Q$$

$$\Leftrightarrow 32R^4+16R^3r+4R^2r^2+8Rr^3+3r^4 \underset{(***)}{\overset{?}{\ge}} (8R^2-4Rr+r^2)s^2$$

$$\text{Finally}, (8R^2-4Rr+r^2)s^2 \overset{\text{Rouche}}{\leq}$$

$$(8R^2-4Rr+r^2)\left(2R^2+10Rr-r^2+2(R-2r).\sqrt{R^2-2Rr}\right) \overset{?}{\leq} \text{LHS of } (***)$$

$$\Leftrightarrow 2(R-2r)(8R^3-12R^2r+Rr^2-r^3) \overset{?}{\geq} 2(R-2r).\sqrt{R^2-2Rr}.(8R^2-4Rr+r^2)$$

$$\text{and} \because R-2r \overset{\text{Euler}}{\geq} 0 \therefore \text{it suffices to prove}:$$

$$(8R^3-12R^2r+Rr^2-r^3) \overset{?}{>} (R^2-2Rr)(8R^2-4Rr+r^2)^2$$

$$\Leftrightarrow r^3(32R^3+8R^2r+r^3) \overset{?}{>} 0 \to \text{trivially true} \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{\text{cyc}} \tan^2 A \geq \sum_{\text{cyc}} \cot^2 \frac{A}{2} \ \forall \text{ acute } \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$