ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{a^3}{{h_a}^3}+\frac{a^3}{{h_b}^3}+\frac{a^3}{{h_c}^3}\ge\frac{a^3}{{r_a}^3}+\frac{a^3}{{r_b}^3}+\frac{a^3}{{r_c}^3}$$

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Solution by Jenish Rijal-Nepal

$$\sum_{cyc}\frac{a^3}{{h_a}^3}\ge\sum_{cyc}\frac{a^3}{{r_a}^3}\Leftrightarrow\sum_{cyc}\frac{a^3}{{h_a}^3}=\sum_{cyc}\frac{a^6}{8\Delta^3}\ge\sum_{cyc}\frac{a^3}{{r_a}^3}=\sum_{cyc}\frac{a^3(b+c-a)^3}{8\Delta^3}\Leftrightarrow$$

$$\sum_{cyc}a^6\ge\sum_{cyc}a^3(b+c-a)^3$$

$Thus, it\ suffices\ to\ prove\ that:\ \sum_{cyc}a^3(b+c-a)^3\le\sum_{cyc}a^6$

$By\ AM-GM\ Inequality,$

$$a(b+c-a)\le\left(\frac{a+b+c-a}{2}\right)^2=\left(\frac{b+c}{2}\right)^2\quad\Rightarrow\ a^3(b+c-a)^3\le\left(\frac{b+c}{2}\right)^6$$

$By\ symmetry,\ the\ same\ holds\ for\ b\ and\ c.$

$$\Rightarrow\sum_{cyc}a^3(b+c-a)^3\ \le\ \sum_{cyc}\left(\frac{b+c}{2}\right)^6\ \overset{Power\ Mean}{\le}\ \sum_{cyc}\left(\frac{b^6+c^6}{2}\right)=\sum_{cyc}a^6.\ \blacksquare$$

$Equality\ holds\ if\ and\ only\ if\ the\ triangle\ is\ equilateral.$