ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{1}{\sqrt{\cos A}} \geq \sum_{cyc}\frac{1}{\sqrt{\sin\frac{A}{2}}}$$

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$$\sum_{cyc}\frac{1}{\sqrt{\sin\frac{A}{2}}} = \sum_{cyc}\frac{\sqrt{2\cos\frac{B-C}{2}}}{\sqrt{2\cos\frac{B+C}{2}\cos\frac{B-C}{2}}} \leq \sum_{cyc}\frac{\sqrt{2}}{\sqrt{2\cos\frac{B+C}{2}\cos\frac{B-C}{2}}}$$

$$\left(\because 0 < \cos\frac{B-C}{2} \leq 1\right) = \sum_{cyc}\frac{\sqrt{2}}{\sqrt{\cos B + \cos C}}$$

$$\therefore \sum_{cyc}\frac{1}{\sqrt{\sin\frac{A}{2}}} \leq \sum_{cyc}\frac{\sqrt{2}}{\sqrt{\cos B + \cos C}} \to (1)$$

$$\text{Again, } \sum_{cyc}\frac{1}{\sqrt{\cos A}} = \frac{1}{2}\cdot\sum_{cyc}\left(\frac{1}{\sqrt{\cos B}} + \frac{1}{\sqrt{\cos C}}\right) \overset{\text{Bergstrom}}{\geq} \frac{1}{2}\cdot\sum_{cyc}\frac{4}{\sqrt{\cos B} + \sqrt{\cos C}} \overset{\text{CBS}}{\geq}$$

$$\sum_{cyc}\frac{2}{\sqrt{2(\cos B + \cos C)}} = \sum_{cyc}\frac{\sqrt{2}}{\sqrt{\cos B + \cos C}} \overset{\text{via }(1)}{\geq} \sum_{cyc}\frac{1}{\sqrt{\sin\frac{A}{2}}} \text{ and so,}$$

$$\sum_{cyc}\frac{1}{\sqrt{\cos A}} \geq \sum_{cyc}\frac{1}{\sqrt{\sin\frac{A}{2}}} \quad \forall \text{ acute } \Delta \text{ ABC}, '' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$