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In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{1}{\sqrt{\sin A}} \ge \sum_{cyc}\frac{1}{\sqrt{\cos\frac{A}{2}}}$$

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$$\sum_{cyc}\frac{1}{\sqrt{\cos\frac{A}{2}}} = \sum_{cyc}\frac{\sqrt{2\cos\frac{B-C}{2}}}{\sqrt{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}} \le \sum_{cyc}\frac{\sqrt{2}}{\sqrt{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}}$$

$$\left(\because 0 < \cos\frac{B-C}{2} \le 1\right) = \sum_{cyc}\frac{\sqrt{2}}{\sqrt{\sin B + \sin C}}$$

$$\therefore \sum_{cyc}\frac{1}{\sqrt{\cos\frac{A}{2}}} \le \sum_{cyc}\frac{\sqrt{2}}{\sqrt{\sin B + \sin C}} \to (1)$$

$$\text{Again, } \sum_{cyc}\frac{1}{\sqrt{\sin A}} = \frac{1}{2}.\sum_{cyc}\left(\frac{1}{\sqrt{\sin B}} + \frac{1}{\sqrt{\sin C}}\right) \overset{\text{Bergstrom}}{\ge} \frac{1}{2}.\sum_{cyc}\frac{4}{\sqrt{\sin B}+\sqrt{\sin C}}$$

$$\overset{\text{CBS}}{\ge} \sum_{cyc}\frac{2}{\sqrt{2(\sin B + \sin C)}} = \sum_{cyc}\frac{\sqrt{2}}{\sqrt{\sin B + \sin C}} \overset{\text{via (1)}}{\ge} \sum_{cyc}\frac{1}{\sqrt{\cos\frac{A}{2}}} \text{ and so,}$$

$$\sum_{cyc}\frac{1}{\sqrt{\sin A}} \ge \sum_{cyc}\frac{1}{\sqrt{\cos\frac{A}{2}}} \ \forall \Delta \text{ ABC}, '' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$