ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\tan A}{\tan B}\geq\sum_{cyc}\frac{\sin 2A}{\sin 2B}$$

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Let us consider a $\Delta\, A'B'C'$ with angles $A'\equiv(\pi-2A), B'\equiv(\pi-2B)$ and $C'\equiv(\pi-2C)$ & then : $\cos A'\cos B'\cos C'=\cos(\pi-2A)\cos(\pi-2B)\cos(\pi-2C)$ $=-\cos 2A\cos 2B\cos 2C=1+4\cos A\cos B\cos C>0$ ($\because \Delta$ ABC being acute $\Rightarrow \cos A\cos B\cos C>0$) $\Rightarrow \Delta\, A'B'C'$ is acute $\rightarrow(*)$

$$\text{Now, }\sum_{cyc}\frac{\cot\frac{A}{2}}{\cot\frac{B}{2}}\overset{?}{\geq}\sum_{cyc}\frac{\sin A}{\sin B}\Leftrightarrow\sum_{cyc}\frac{r_b}{r_a}\overset{?}{\geq}\sum_{cyc}\frac{a}{b}\Leftrightarrow\sum_{cyc}\frac{s-a}{s-b}\overset{?}{\geq}\sum_{cyc}\frac{a}{b}$$

$$\Leftrightarrow\sum_{cyc}\frac{x}{y}\overset{?}{\geq}\sum_{cyc}\frac{y+z}{z+x}\begin{pmatrix}x=s-a,y=s-b,z=s-c\Rightarrow\\ a=y+z,b=z+x,c=x+y\end{pmatrix}$$

$$\Leftrightarrow\frac{1}{xyz}\cdot\sum_{cyc}zx^2\overset{?}{\geq}\frac{1}{(x+y)(y+z)(z+x)}\cdot\sum_{cyc}\left((x+y)(y+z)^2\right)$$

$$\Leftrightarrow\sum_{cyc}x^2y^4+\sum_{cyc}x^3y^3\underset{①}{\overset{?}{\geq}}xyz\sum_{cyc}x^2y+3x^2y^2z^2$$

$$\text{Now, }\sum_{cyc}x^2y^4=(xy^2)^2+(yz^2)^2+(zx^2)^2\geq$$

$$(xy^2)(yz^2)+(yz^2)(zx^2)+(zx^2)(xy^2)=xyz\sum_{cyc}x^2y\text{ and also,}$$

$$\sum_{cyc}x^3y^3\overset{AM-GM}{\geq}3x^2y^2z^2\text{ and so, }\sum_{cyc}x^2y^4+\sum_{cyc}x^3y^3\geq xyz\sum_{cyc}x^2y+3x^2y^2z^2$$

$$\Rightarrow ①\text{ is true }\therefore\sum_{cyc}\frac{\cot\frac{A}{2}}{\cot\frac{B}{2}}\geq\sum_{cyc}\frac{\sin A}{\sin B}\ \forall\,\Delta\text{ ABC and implementing it}$$

$$\text{on }\Delta\,A'B'C'\text{, we get : }\sum_{cyc}\frac{\cot\frac{\pi-2A}{2}}{\cot\frac{\pi-2B}{2}}\geq\sum_{cyc}\frac{\sin(\pi-2A)}{\sin(\pi-2B)}\text{ on }\Delta\,A'B'C'\text{, we get :}$$

$$\sum_{cyc}\frac{\cot\frac{\pi-2A}{2}}{\cot\frac{\pi-2B}{2}}\geq\sum_{cyc}\frac{\sin(\pi-2A)}{\sin(\pi-2B)}\Rightarrow\sum_{cyc}\frac{\tan A}{\tan B}\geq\sum_{cyc}\frac{\sin 2A}{\sin 2B}\text{ and }\because$$

$$\Delta\,A'B'C'\overset{\text{via }(*)}{\text{ is }}\text{ acute, hence : }\sum_{cyc}\frac{\tan A}{\tan B}\geq\sum_{cyc}\frac{\sin 2A}{\sin 2B}\ \forall\text{ acute }\Delta\text{ ABC,}$$

$''=''$ iff Δ ABC is equilateral (QED)