ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\cot\frac{A}{2}}{\cot\frac{B}{2}}\ge\sum_{cyc}\frac{\sin A}{\sin B}$$

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$$\sum_{cyc}\frac{\cot\frac{A}{2}}{\cot\frac{B}{2}}\overset{?}{\ge}\sum_{cyc}\frac{\sin A}{\sin B}\Leftrightarrow\sum_{cyc}\frac{r_b}{r_a}\overset{?}{\ge}\sum_{cyc}\frac{a}{b}\Leftrightarrow\sum_{cyc}\frac{s-a}{s-b}\overset{?}{\ge}\sum_{cyc}\frac{a}{b}$$

$$\Leftrightarrow\sum_{cyc}\frac{x}{y}\overset{?}{\ge}\sum_{cyc}\frac{y+z}{z+x}\ \begin{pmatrix}x=s-a,y=s-b,z=s-c\Rightarrow\\ a=y+z,b=z+x,c=x+y\end{pmatrix}$$

$$\Leftrightarrow\frac{1}{xyz}.\sum_{cyc}zx^2\overset{?}{\ge}\frac{1}{(x+y)(y+z)(z+x)}.\sum_{cyc}\left((x+y)(y+z)^2\right)$$

$$\Leftrightarrow\sum_{cyc}x^2y^4+\sum_{cyc}x^3y^3\underset{①}{\overset{?}{\ge}}xyz\sum_{cyc}x^2y+3x^2y^2z^2$$

$$\text{Now,}\sum_{cyc}x^2y^4=(xy^2)^2+(yz^2)^2+(zx^2)^2\ge$$

$$(xy^2)(yz^2)+(yz^2)(zx^2)+(zx^2)(xy^2)=xyz\sum_{cyc}x^2y\ \text{ and also,}$$

$$\sum_{cyc}x^3y^3\overset{AM-GM}{\ge}3x^2y^2z^2\text{ and so,}\sum_{cyc}x^2y^4+\sum_{cyc}x^3y^3\ge xyz\sum_{cyc}x^2y+3x^2y^2z^2$$

$$\Rightarrow ①\text{ is true}\therefore\sum_{cyc}\frac{\cot\frac{A}{2}}{\cot\frac{B}{2}}\ge\sum_{cyc}\frac{\sin A}{\sin B}\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$