ROMANIAN MATHEMATICAL MAGAZINE

If in any Δ ABC the following relationship holds :

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\geq 1+\frac{r}{R}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}\sin\frac{A}{2}\right)^2=\sum_{cyc}\sin^2\frac{A}{2}+2\sum_{cyc}\left(\sin\frac{B}{2}\sin\frac{C}{2}\right)=\frac{2R-r}{2R}+\frac{2r}{4R}.\sum_{cyc}\csc\frac{A}{2}$$

$$\overset{\text{Jensen}}{\geq}\frac{2R-r}{2R}+\frac{r}{2R}.3\csc\frac{\pi}{6}\left(\because f(x)=\csc\frac{x}{2}\ \forall\ x\in(0,\pi)\text{ is convex}\right)=\frac{2R-r}{2R}+\frac{6r}{2R}$$

$$=\frac{2R+5r}{2R}\overset{?}{\geq}\frac{(R+r)^2}{R^2}\Leftrightarrow 5Rr\overset{?}{\geq}4Rr+2r^2\Leftrightarrow R\overset{?}{\geq}2r\rightarrow\text{true via Euler}$$

$$\therefore\left(\sum_{cyc}\sin\frac{A}{2}\right)^2\geq\left(1+\frac{r}{R}\right)^2\Rightarrow\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\geq 1+\frac{r}{R}\ \forall\ \Delta\ ABC,$$

$''=''$ iff Δ ABC is equilateral (QED)