

# ROMANIAN MATHEMATICAL MAGAZINE

In  $\triangle ABC$  the following relationship holds:

$$\frac{h_a + h_b}{a} + \frac{h_b + h_c}{b} + \frac{h_c + h_a}{c} \geq 6\sqrt{3} \cdot \frac{r}{R}$$

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Lemma: In  $\triangle ABC$  holds:

$$h_a h_b h_c \geq 27r^3$$

Proof:

$$\begin{aligned} h_a h_b h_c &= \frac{8F^3}{abc} = \frac{8F^3}{4RF} = \frac{2F^2}{R} = \frac{2r^2 s^2}{R} \\ \frac{2r^2 s^2}{R} &\geq 27r^3 \Leftrightarrow \frac{2s^2}{R} \geq 27r \Leftrightarrow 2s^2 \geq 27Rr \\ 2s^2 &\stackrel{\text{GERRETSEN}}{\geq} 2(16Rr - 5r^2) \geq 27Rr \\ 32Rr - 10r^2 &\geq 27Rr \Leftrightarrow R \geq 2r \quad (\text{Euler}) \end{aligned}$$

Back to the problem:

$$\begin{aligned} \sum_{cyc} \frac{h_a + h_b}{a} &\stackrel{\text{AM-GM}}{\geq} 3 \sqrt[3]{\frac{(h_a + h_b)(h_b + h_c)(h_c + h_a)}{abc}} \geq \\ &\stackrel{\text{CESARO}}{\geq} 3 \sqrt[3]{\frac{8h_a h_b h_c}{4RF}} \stackrel{\text{Lemma}}{\geq} 3 \sqrt[3]{\frac{8 \cdot 27r^3}{4Rrs}} = 3 \cdot 2 \cdot 3r \cdot \sqrt[3]{\frac{1}{4rs}} \stackrel{\text{EULER}}{\geq} \\ &\geq \frac{18r}{\sqrt[3]{4R \cdot \frac{R}{2} \cdot s}} \stackrel{\text{MITRINOVIC}}{\geq} \frac{18r}{\sqrt[3]{2R^2 s}} \geq \\ &\geq \frac{18r}{\sqrt[3]{2R^2 \cdot \frac{3\sqrt{3}}{2} \cdot R}} = \frac{18r}{\sqrt[3]{(R\sqrt{3})^3}} = \frac{18r}{\sqrt{3}R} = \frac{18\sqrt{3}r}{3R} = 6\sqrt{3} \cdot \frac{r}{R} \end{aligned}$$

Equality holds for  $a = b = c$ .