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In any Δ ABC the following relationship holds :

$$\sum_{cyc} \frac{a+b}{(\cos A+\cos B)^2} \geq 6R\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{b+c}{(\cos B+\cos C)^2} = \sum_{cyc} \frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{4\sin^2\frac{A}{2}.\cos^2\frac{B-C}{2}} = R.\sum_{cyc} \frac{\cos\frac{A}{2}}{\sin^2\frac{A}{2}.\cos\frac{B-C}{2}} \geq$$

$$R.\sum_{cyc} \frac{\cos\frac{A}{2}}{\sin^2\frac{A}{2}} \left(\because \cos\frac{B-C}{2} \leq 1\right) \overset{AM-GM}{\geq} 3R.\sqrt[3]{\prod_{cyc} \frac{\cos\frac{A}{2}}{\sin^2\frac{A}{2}}} = 3R.\sqrt[3]{\frac{s}{4R}.\frac{16R^2}{r^2}}$$

$$= 3R.\sqrt[3]{\frac{4Rs}{r^2}} \overset{\substack{Euler \\ and \\ Mitrinovic}}{\geq} 3R.\sqrt[3]{\frac{8r.3\sqrt{3}.r}{r^2}} = 6R.\sqrt{3} \text{ and so,}$$

$$\sum_{cyc} \frac{a+b}{(\cos A+\cos B)^2} \geq 6R.\sqrt{3}\ \forall\ \Delta\ ABC, '' = ''\text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$