ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{a^3+b^3}{h_c^2}\le\sum_{cyc}\frac{a^3+b^3}{r_c^2}$$

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WLOG we may assume : $a\ge b\ge c$ and then :

$$b^3+c^3\le c^3+a^3\le a^3+b^3,\ \frac{1}{h_a^2}\ge\frac{1}{h_b^2}\ge\frac{1}{h_b^2}\ \text{and}\ \frac{1}{r_a^2}\le\frac{1}{r_b^2}\le\frac{1}{r_b^2}\ \text{and}$$

$$\text{hence via Chebyshev, }\sum_{cyc}\frac{b^3+c^3}{h_a^2}-\sum_{cyc}\frac{b^3+c^3}{r_a^2}\le$$

$$\frac{1}{3}\cdot\left(\sum_{cyc}(b^3+c^3)\right)\left(\sum_{cyc}\frac{1}{h_a^2}\right)-\frac{1}{3}\cdot\left(\sum_{cyc}(b^3+c^3)\right)\left(\sum_{cyc}\frac{1}{r_a^2}\right)$$

$$=\frac{1}{3}\cdot\left(\sum_{cyc}(b^3+c^3)\right)\left(\frac{1}{4r^2s^2}\left(\sum_{cyc}a^2-4\sum_{cyc}(s-a)^2\right)\right)$$

$$=\frac{1}{6r^2s^2}\cdot\left(\sum_{cyc}(b^3+c^3)\right)\left(s^2-4Rr-r^2-2\left(3s^2-4s^2+2(s^2-4Rr-r^2)\right)\right)$$

$$=\frac{-1}{6r^2s^2}\cdot\left(\sum_{cyc}(b^3+c^3)\right)(s^2-12Rr-3r^2)\le 0\ \because\ s^2-12Rr-3r^2$$

$$=s^2-16Rr+5r^2+4r(R-2r)\overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Euler}}}{\ge}0\ \therefore\ \sum_{cyc}\frac{a^3+b^3}{h_c^2}\le\sum_{cyc}\frac{a^3+b^3}{r_c^2}\ \forall\,\Delta\,\text{ABC},$$

$''=''$ iff Δ ABC is equilateral (QED)