ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC the following relationship holds :

$$\frac{a}{\cos^3 B}+\frac{b}{\cos^3 C}+\frac{c}{\cos^3 A}\geq 24\sqrt{3}R$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let us consider a Δ A′B′C′ with angles $A'\equiv(\pi-2A), B'\equiv(\pi-2B)$ and $C'\equiv(\pi-2C)$ and then : $\cos A'\cos B'\cos C' =$

$\cos(\pi-2A)\cos(\pi-2B)\cos(\pi-2C) = -\cos 2A\cos 2B\cos 2C$

$= 1+4\cos A\cos B\cos C > 0$ ($\because \Delta$ ABC being acute $\Rightarrow \cos A\cos B\cos C > 0$)

$$\Rightarrow \Delta A'B'C' \text{ is acute} \to (1) \text{ and now, } \sum_{cyc}\frac{\cos\frac{A}{2}}{\sin^3\frac{B}{2}} \overset{AM-GM}{\geq} 3.\sqrt[3]{\prod_{cyc}\frac{\cos\frac{A}{2}}{\sin^3\frac{B}{2}}} = 3.\sqrt[3]{\frac{\left(\frac{s}{4R}\right)}{\left(\frac{r^3}{64R^3}\right)}}$$

$$= 3.\sqrt[3]{\frac{16R^2s}{r^3}} \overset{\substack{Euler\\ and\\ Mitrinovic}}{\geq} 3.\sqrt[3]{\frac{64r^2.3\sqrt{3}.r}{r^3}} \Rightarrow \sum_{cyc}\frac{\cos\frac{A}{2}}{\sin^3\frac{B}{2}} \geq 12\sqrt{3} \text{ and applying it on}$$

$$\Delta A'B'C', \text{we get : } \sum_{cyc}\frac{\cos\frac{\pi-2A}{2}}{\sin^3\frac{\pi-2B}{2}} \geq 12\sqrt{3} \Rightarrow \sum_{cyc}\frac{\sin A}{\cos^3 B} \geq 12\sqrt{3}$$

$$\Rightarrow \sum_{cyc}\frac{\frac{a}{2R}}{\cos^3 B} \geq 12\sqrt{3} \Rightarrow \sum_{cyc}\frac{a}{\cos^3 B} \geq 24\sqrt{3}.R \text{ and } \because \Delta A'B'C' \overset{via (1)}{\text{ is }} \text{ acute,}$$

$$\therefore \frac{a}{\cos^3 B}+\frac{b}{\cos^3 C}+\frac{c}{\cos^3 A} \geq 24\sqrt{3}.R \ \forall \text{ acute } \Delta ABC,$$

$'' = ''$ iff Δ ABC is equilateral (QED)