ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{a+b}{(\cos A+\cos B)^3}\ge 6R\sqrt{3}$$

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$$\sum_{cyc}\frac{b+c}{(\cos B+\cos C)^3}=\sum_{cyc}\frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{8\sin^3\frac{A}{2}.\cos^3\frac{B-C}{2}}=\frac{R}{2}.\sum_{cyc}\frac{\cos\frac{A}{2}}{\sin^3\frac{A}{2}.\cos^2\frac{B-C}{2}}\ge$$

$$\frac{R}{2}.\sum_{cyc}\frac{\cos\frac{A}{2}}{\sin^3\frac{A}{2}}\left(\because \cos^2\frac{B-C}{2}\le 1\right)=\frac{Rs}{2r^2}.\sum_{cyc}\left(\frac{1}{r_a}.\frac{r^2}{\sin^2\frac{A}{2}}\right)$$

$$=\frac{Rs}{2r^2.rs^2}.\sum_{cyc}\left(s(s-a).AI^2\right)=\frac{R}{2r^3}.\sum_{cyc}\left((s-a).(bc-4Rr)\right)$$

$$=\frac{R}{2r^3}.\left(s(s^2+4Rr+r^2)-12Rrs-4Rrs\right)=\frac{Rs(s^2-12Rr+r^2)}{2r^3}\overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Mitrinovic}}}{\ge}$$

$$\frac{R.3\sqrt{3}r.(4Rr-4r^2)}{2r^3}=\frac{R.3\sqrt{3}.(2R-2r)}{r}\overset{\text{Euler}}{\ge}\frac{R.3\sqrt{3}.(2r)}{r}=6R.\sqrt{3}\text{ and so,}$$

$$\sum_{cyc}\frac{a+b}{(\cos A+\cos B)^3}\ge 6R.\sqrt{3}\ \forall\ \Delta\ ABC,''=''\text{ iff }\Delta\ ABC\text{ is equilateral (QED)}$$