ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{1}{1+\tan^4\frac{A}{2}}+\frac{1}{1+\tan^4\frac{B}{2}}+\frac{1}{1+\tan^4\frac{C}{2}}\le\frac{27}{10}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{1+\tan^4\frac{A}{2}}=\sum_{cyc}\frac{1+\tan^4\frac{A}{2}-\tan^4\frac{A}{2}}{1+\tan^4\frac{A}{2}}=3-\sum_{cyc}\frac{r_a^4}{s^4+r_a^4}\overset{\text{Bergstrom}}{\le}$$

$$3-\frac{\left(\sum_{cyc}r_a^2\right)^2}{3s^4+\sum_{cyc}r_a^4}=3-\frac{((4R+r)^2-2s^2)^2}{3s^4+((4R+r)^2-2s^2)^2-2(s^4-2r(4R+r)s^2)}$$

$$\Leftrightarrow 25s^4-(448R^2+272Rr+40r^2)s^2+7(4R+r)^4\underbrace{\overset{?}{\ge}}_{(*)}0$$

$$\text{Now, since } 25s^2(s^2-16Rr+5r^2)\overset{\text{Gerretsen}}{\ge}0\ \therefore \text{ in order to prove } (*),$$

$$\text{it suffices to prove : LHS of } (*)\overset{?}{\ge}25s^2(s^2-16Rr+5r^2)$$

$$\Leftrightarrow 7(4R+r)^4\underbrace{\overset{?}{\ge}}_{(**)}(448R^2-128Rr+165r^2)s^2 \text{ and again,}$$

$$(448R^2-128Rr+165r^2)s^2\overset{\text{Gerretsen}}{\le}$$

$$(448R^2-128Rr+165r^2)(4R^2+4Rr+3r^2)\overset{?}{\le}7(4R+r)^4$$

$$\Leftrightarrow 128t^3-205t^2-41t-122\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(128t^2+51t+61)\overset{?}{\ge}0$$

$$\to \text{true} \because t\overset{\text{Euler}}{\ge}2\Rightarrow(**)\Rightarrow(*) \text{ is true }\therefore \frac{1}{1+\tan^4\frac{A}{2}}+\frac{1}{1+\tan^4\frac{B}{2}}+\frac{1}{1+\tan^4\frac{C}{2}}$$

$$\le\frac{27}{10}\ \forall\,\Delta\,ABC,\text{"="} \text{ iff } \Delta\,ABC \text{ is equilateral}$$