ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{a^2}{h_a^3}+\frac{b^2}{h_b^3}+\frac{c^2}{h_c^3}\le\frac{a^2}{r_a^3}+\frac{b^2}{r_b^3}+\frac{c^2}{r_c^3}$$

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$$\sum_{cyc}\frac{a^2}{h_a^3}\overset{?}{\ge}\sum_{cyc}\frac{a^2}{r_a^3}\Leftrightarrow\sum_{cyc}\frac{a^5}{8r^3s^3}\overset{?}{\ge}\sum_{cyc}\frac{a^2(s-a)^3}{r^3s^3}$$

$$\Leftrightarrow\sum_{cyc}a^5\overset{?}{\ge}8\sum_{cyc}\left(a^2(s^3-3s^2a+3sa^2-a^3)\right)$$

$$\Leftrightarrow 9\left(\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^3\right)-2s\sum_{cyc}a^2b^2+abc\sum_{cyc}ab\right)\overset{?}{\ge}$$

$$8s^3\left(\sum_{cyc}a^2\right)-24s^2\left(\sum_{cyc}a^3\right)+24s\left(2\sum_{cyc}a^2b^2-16r^2s^2\right)$$

$$\Leftrightarrow 9\begin{pmatrix}4s(s^2-4Rr-r^2)(s^2-6Rr-3r^2)-2s((s^2+4Rr+r^2)^2-16Rrs^2)\\+8Rrs(s^2+4Rr+r^2)\end{pmatrix}\overset{?}{\ge}$$

$$16s^3(s^2-4Rr-r^2)-48s^3(s^2-6Rr-3r^2)+$$
$$48s((s^2+4Rr+r^2)^2-16Rrs^2-8r^2s^2)$$

$$\Leftrightarrow s^4-(10Rr+10r^2)s^2-r^2(24R^2-78Rr-21r^2)\boxed{\underset{①}{\overset{?}{\ge}}}0 \text{ and } \because$$

$$(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\ge}0\ \therefore \text{ in order to prove ①, it suffices to prove :}$$

$$\text{LHS of ①}\overset{?}{\ge}(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow(11R-10r)s^2\boxed{\underset{②}{\overset{?}{\ge}}}r(140R^2-119Rr+2r^2)$$

$$\text{Again, }(11R-10r)s^2\overset{\text{Gerretsen}}{\ge}(11R-10r)(16Rr-5r^2)\overset{?}{\ge}$$

$$r(140R^2-119Rr+2r^2)\Leftrightarrow 12r(R-2r)(3R-2r)\overset{?}{\ge}0\to\text{true}\because R\overset{\text{Euler}}{\ge}2r$$

$$\Rightarrow ②\Rightarrow ① \text{ is true }\forall\,\Delta\,ABC\ \therefore\frac{a^2}{h_a^3}+\frac{b^2}{h_b^3}+\frac{c^2}{h_c^3}\le\frac{a^2}{r_a^3}+\frac{b^2}{r_b^3}+\frac{c^2}{r_c^3}\ \forall\,\Delta\,ABC,$$

'' = '' iff Δ ABC is equilateral (QED)