ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{a^3}{h_a^3}+\frac{b^3}{h_b^3}+\frac{c^3}{h_c^3}\le\frac{a^3}{r_a^3}+\frac{b^3}{r_b^3}+\frac{c^3}{r_c^3}$$

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$$\sum_{cyc}\frac{a^3}{h_a^3}\overset{?}{\ge}\sum_{cyc}\frac{a^3}{r_a^3}\Leftrightarrow\sum_{cyc}\frac{a^6}{8r^3s^3}\overset{?}{\ge}\sum_{cyc}\frac{a^3(s-a)^3}{r^3s^3}$$

$$\Leftrightarrow\sum_{cyc}a^6\overset{?}{\ge}8\sum_{cyc}\left(a^3(s^3-3s^2a+3sa^2-a^3)\right)$$

$$\Leftrightarrow 9\left(\left(\sum_{cyc}a^3\right)^2-2\sum_{cyc}a^3b^3\right)\overset{?}{\ge}8s^3\left(\sum_{cyc}a^3\right)-24s^2\left(2\sum_{cyc}a^2b^2-16r^2s^2\right)+$$

$$24s\left(\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^3\right)-2s\sum_{cyc}a^2b^2+abc\sum_{cyc}ab\right)$$

$$\Leftrightarrow 9\left(4s^2(s^2-6Rr-3r^2)^2-2\left((s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\right)\right)\overset{?}{\ge}$$

$$16s^4(s^2-6Rr-3r^2)-48s^2((s^2+4Rr+r^2)^2-16Rrs^2-8r^2s^2)+$$

$$24s\begin{pmatrix}4s(s^2-4Rr-r^2)(s^2-6Rr-3r^2)-2s((s^2+4Rr+r^2)^2-16Rrs^2)\\+8Rrs(s^2+4Rr+r^2)\end{pmatrix}$$

$$\Leftrightarrow s^6-(12Rr+15r^2)s^4-r^2(72R^2+120Rr+39r^2)s^2-9r^3(4R+r)^3\overset{?}{\ge}0\ (1)\ \text{and}$$

$$\because (s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\ge}0\ \therefore\ \text{in order to prove (1), it suffices to prove :}$$

$$\text{LHS of (1)}\overset{?}{\ge}(s^2-16Rr+5r^2)^3\Leftrightarrow(18R-15r)s^4-$$

$$r(348R^2-300Rr+18r^2)s^2-r^3(2136R^2-546Rr+67r^2)\overset{?}{\ge}0\ (2)$$

$$\text{Again, since }(18R-15r)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\ge}0\ \therefore\ \text{in order to prove (2),}$$

$$\text{it suffices to prove : LHS of (2)}\overset{?}{\ge}(18R-15r)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow(57R^2-90Rr+33r^2)s^2\overset{?}{\ge}r(712R^3-1146R^2r+576Rr^2-77r^3)\ (3)$$

$$\text{Finally, }(57R^2-90Rr+33r^2)s^2\overset{\text{Gerretsen}}{\ge}(57R^2-90Rr+33r^2)(16Rr-5r^2)$$

$$\overset{?}{\ge}r(712R^3-1146R^2r+576Rr^2-77r^3)\Leftrightarrow 200t^3-579t^2+402t-88\overset{?}{\ge}0$$

$$\left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(200t^2-179t+44)\overset{?}{\ge}0\to\text{true}\because t\overset{\text{Euler}}{\ge}2$$

$$\Rightarrow ③ \Rightarrow ② \Rightarrow ① \text{ is true } \forall \, \Delta \, ABC \therefore \frac{a^3}{h_a^3} + \frac{b^3}{h_b^3} + \frac{c^3}{h_c^3} \leq \frac{a^3}{r_a^3} + \frac{b^3}{r_b^3} + \frac{c^3}{r_c^3} \; \forall \, \Delta \, ABC,$$

$$'' = '' \text{ iff } \Delta \, ABC \text{ is equilateral } (QED)$$