ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$h_a^3 + h_b^3 + h_c^3 \le r_a^3 + r_b^3 + r_c^3$$

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$$\sum_{cyc} r_a^3 \overset{?}{\ge} \sum_{cyc} h_a^3$$

$$\Leftrightarrow (4R+r)^3 - 12Rs^2 \overset{?}{\ge} \frac{(s^2+4Rr+r^2)^3 - 24Rrs^2(s^2+2Rr+r^2)}{8R^3}$$

$$\Leftrightarrow -s^6 + (12Rr - 3r^2)s^4 - (96R^4 + 3r^4)s^2 +$$

$$512R^6 + 384R^5r + 96R^4r^2 - 56R^3r^3 - 48R^2r^4 - 12Rr^5 - r^6 \underset{①}{\underbrace{\overset{?}{\ge}}} 0$$

$$\text{Now, since : } P = -s^4(s^2 - 4R^2 - 4Rr - 3r^2) \overset{\text{Gerretsen}}{\ge} 0$$

$$\therefore \text{ in order to prove ①, it suffices to prove : LHS of ① } \overset{?}{\ge} P$$

$$\Leftrightarrow -(4R^2 - 8Rr + 6r^2)s^4 - (96R^4 + 3r^4)s^2 +$$

$$512R^6 + 384R^5r + 96R^4r^2 - 56R^3r^3 - 48R^2r^4 - 12Rr^5 - r^6 \underset{②}{\underbrace{\overset{?}{\ge}}} 0$$

$$\text{Again, since : } Q = -(4R^2 - 8Rr + 6r^2)(s^2 - 4R^2 - 4Rr - 3r^2)s^2 \overset{\text{Gerretsen}}{\ge} 0$$

$$\therefore \text{ in order to prove ②, it suffices to prove : LHS of ② } \overset{?}{\ge} Q$$

$$\Leftrightarrow 512R^6 + 384R^5r + 96R^4r^2 - 56R^3r^3 - 48R^2r^4 - 12Rr^5 - r^6$$

$$\underset{③}{\underbrace{\overset{?}{\ge}}} (112R^4 - 16R^2r^2 + 4Rr^3 + 21r^4)s^2$$

$$\text{Finally, } (112R^4 - 16R^2r^2 + 4Rr^3 + 21r^4)s^2 \overset{\text{Gerretsen}}{\le}$$

$$(112R^4 - 16R^2r^2 + 4Rr^3 + 21r^4)(4R^2 + 4Rr + 3r^2) \overset{?}{\le} \text{LHS of ③}$$

$$\Leftrightarrow 8t^6 - 24t^4 - 3t^3 - 18t^2 - 12t - 8 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(8t^5 + 16t^4 + 8t^3 + 13t^2 + 8t + 4\right) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\Rightarrow ③ \Rightarrow ② \Rightarrow ① \text{ is true } \therefore h_a^3 + h_b^3 + h_c^3 \le r_a^3 + r_b^3 + r_c^3 \ \forall \ \Delta ABC,$$

$$'' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$