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In $\triangle ABC$ the following relationship holds:

$$\frac{\tan(A) + \tan(B)}{\tan^2(C)} + \frac{\tan(B) + \tan(C)}{\tan^2(A)} + \frac{\tan(A) + \tan(C)}{\tan^2(B)} \geq 2\sqrt{3}$$

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Solution by Mirsadix Muzefferov-Azerbaijan

$$\begin{aligned} \sum_{cyc} \frac{\tan(A) + \tan(B)}{\tan^2(C)} &= \sum_{cyc} \frac{\sin(A+B)}{\cos(A) \cdot \cos(B) \cdot \tan^2(C)} = \sum_{cyc} \frac{\cos^2(C) \cdot \sin(C)}{\sin^2(C) \cdot \cos(A) \cdot \cos(B)} = \\ &= \sum_{cyc} \frac{\cos^2(C)}{\sin(C) \cdot \cos(A) \cdot \cos(B)} \stackrel{AM-GM}{\geq} \\ &\geq 3 \left(\frac{\cos^2(A) \cdot \cos^2(B) \cdot \cos^2(C)}{\sin(A) \cdot \sin(B) \cdot \sin(C) \cdot \cos^2(A) \cdot \cos^2(B) \cdot \cos^2(C)} \right)^{\frac{1}{3}} = \\ &= 3 \left(\frac{1}{\prod_{cyc} \sin(A)} \right)^{\frac{1}{3}} \geq 3 \left(\frac{1}{\frac{3\sqrt{3}}{8}} \right)^{\frac{1}{3}} = 3 \left(\frac{8}{\sqrt{27}} \right)^{\frac{1}{3}} = 2\sqrt{3} \end{aligned}$$

Equality holds for $a = b = c$.