ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{r_b\sqrt{r_c}}{{r_a}^2}\ge\sqrt{\frac{3}{r}}$$

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$$Let\ r_a=\frac{1}{x},r_b=\frac{1}{y},\ and\ r_c=\frac{1}{z}.$$

$$Using\ the\ identity\sum_{cyc}\frac{1}{r_a}=\frac{1}{r},\ we\ have\ x+y+z=\frac{1}{r}.$$

$$Substituting\ these\ into\ the\ LHS:\ \ LHS=\sum_{cyc}\frac{r_b\sqrt{r_c}}{{r_a}^2}=\sum_{cyc}\frac{\frac{1}{y}\sqrt{\frac{1}{z}}}{\frac{1}{x^2}}=\sum_{cyc}\frac{x^2}{y\sqrt{z}}$$

$$Now,\quad \sum_{cyc}\frac{x^2}{y\sqrt{z}}\overbrace{\ge}^{BERGSTROM}\frac{(x+y+z)^2}{\sum_{cyc}y\sqrt{z}}\mapsto (1)$$

$$Again\ by\ Cauchy's\ Inequality,$$

$$\left(\sum_{cyc}\sqrt{y}\cdot\sqrt{yz}\right)^2=\left(\sum_{cyc}y\sqrt{z}\right)^2\le(x+y+z)(xy+yz+zx)\Rightarrow$$

$$\sum_{cyc}y\sqrt{z}\le\sqrt{(x+y+z)(xy+yz+zx)}$$

$$Substituting\ this\ bound\ back\ into\ (1):$$

$$\sum_{cyc}\frac{x^2}{y\sqrt{z}}\ge\frac{(x+y+z)^2}{\sqrt{(x+y+z)(xy+yz+zx)}}=\sqrt{\frac{(x+y+z)^3}{(xy+yz+zx)}}$$

$$Since\ (a+b+c)^2\ge3(ab+bc+ac),\ it\ follows\ that:$$

$$\sum_{cyc}\frac{h_b\sqrt{h_c}}{{h_a}^2}=\sum_{cyc}\frac{x^2}{y\sqrt{z}}\ge\sqrt{\frac{(x+y+z)^3}{(xy+yz+zx)}}\ge\sqrt{3(x+y+z)}=\sqrt{3\cdot\frac{1}{r}}$$

$$\therefore\sum_{cyc}\frac{r_b\sqrt{r_c}}{{r_a}^2}\ge\sqrt{\frac{3}{r}}$$

$$Equality\ holds\ if\ and\ only\ if\ the\ triangle\ is\ equilateral.$$