

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$10 \sum_{\text{cyc}} \left(\frac{r}{r_a} \right)^3 - 9 \sum_{\text{cyc}} \left(\frac{r}{r_a} \right)^5 \geq 1$$

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Let $x = \frac{3r}{r_a}, y = \frac{3r}{r_b}, z = \frac{3r}{r_c}$ and then : $\sum_{\text{cyc}} x = 3r \cdot \sum_{\text{cyc}} \frac{1}{r_a} = \frac{3r}{r} = 3$ and so,

$$10 \sum_{\text{cyc}} \left(\frac{r}{r_a} \right)^3 - 9 \sum_{\text{cyc}} \left(\frac{r}{r_a} \right)^5 \stackrel{?}{\geq} 1$$

$$\Leftrightarrow \frac{10}{243} \left(\sum_{\text{cyc}} x^3 \right) \left(\sum_{\text{cyc}} x \right)^2 - \frac{9}{243} \left(\sum_{\text{cyc}} x^5 \right) \stackrel{?}{\geq} \frac{1}{243} \left(\sum_{\text{cyc}} x \right)^5$$

$$\Leftrightarrow \sum_{\text{cyc}} x^4 y + \sum_{\text{cyc}} xy^4 \stackrel{?}{\geq} 2xyz \sum_{\text{cyc}} xy \rightarrow \text{true} \because \sum_{\text{cyc}} x^4 y + \sum_{\text{cyc}} xy^4 = \sum_{\text{cyc}} (z(x^4 + y^4))$$

$$\stackrel{\text{AM-GM}}{\geq} 2 \sum_{\text{cyc}} (zx^2y^2) = 2xyz \sum_{\text{cyc}} xy \therefore 10 \sum_{\text{cyc}} \left(\frac{r}{r_a} \right)^3 - 9 \sum_{\text{cyc}} \left(\frac{r}{r_a} \right)^5 \geq 1 \forall \Delta ABC,$$

" = " iff ΔABC is equilateral (QED)