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In any Δ ABC the following relationship holds :

$$10\sum_{cyc}\left(\frac{r}{h_a}\right)^3-9\sum_{cyc}\left(\frac{r}{h_a}\right)^5\ge 1$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x=\frac{3r}{h_a},y=\frac{3r}{h_b},z=\frac{3r}{h_c}\text{ and then : }\sum_{cyc}x=3r.\sum_{cyc}\frac{1}{h_a}=\frac{3r}{r}=3\text{ and so,}$$

$$10\sum_{cyc}\left(\frac{r}{h_a}\right)^3-9\sum_{cyc}\left(\frac{r}{h_a}\right)^5\overset{?}{\ge}1$$

$$\Leftrightarrow\frac{10}{243}\left(\sum_{cyc}x^3\right)\left(\sum_{cyc}x\right)^2-\frac{9}{243}\left(\sum_{cyc}x^5\right)\overset{?}{\ge}\frac{1}{243}\left(\sum_{cyc}x\right)^5$$

$$\Leftrightarrow\sum_{cyc}x^4y+\sum_{cyc}xy^4\overset{?}{\ge}2xyz\sum_{cyc}xy\to\text{true}\because\sum_{cyc}x^4y+\sum_{cyc}xy^4=\sum_{cyc}\left(z(x^4+y^4)\right)$$

$$\overset{AM-GM}{\ge}2\sum_{cyc}(zx^2y^2)=2xyz\sum_{cyc}xy\therefore 10\sum_{cyc}\left(\frac{r}{h_a}\right)^3-9\sum_{cyc}\left(\frac{r}{h_a}\right)^5\ge 1\ \forall\ \Delta\text{ ABC},$$

$$''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$