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In any ΔABC the following relationship holds:

$$\sum_{\text{cyc}} \frac{1}{r_a \cdot \sqrt{\frac{1}{r_b} + \frac{1}{r_c}}} \geq \sqrt{\frac{3}{2r}}$$

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$$\begin{aligned} \sum_{\text{cyc}} \frac{1}{r_a \cdot \sqrt{\frac{1}{r_b} + \frac{1}{r_c}}} &\stackrel{?}{\geq} \sqrt{\frac{3}{2r}} \Leftrightarrow \sum_{\text{cyc}} \frac{1}{\frac{3r}{x} \cdot \sqrt{\frac{y+z}{3r}}} \stackrel{?}{\geq} \sqrt{\frac{3}{2r}} \left(x = \frac{3r}{r_a}, y = \frac{3r}{r_b}, z = \frac{3r}{r_c} \right) \\ &\Leftrightarrow \sum_{\text{cyc}} \frac{x}{\sqrt{y+z}} \stackrel{?}{\geq} \frac{3}{\sqrt{2}} \quad (*) \end{aligned}$$

$$\begin{aligned} \text{Indeed, } \sum_{\text{cyc}} \frac{x}{\sqrt{y+z}} &= \sum_{\text{cyc}} \frac{x^2}{x \cdot \sqrt{y+z}} \stackrel{\text{Bergstrom}}{\geq} \frac{(\sum_{\text{cyc}} x)^2}{\sum_{\text{cyc}} (\sqrt{x} \cdot \sqrt{xy+zx})} \stackrel{\text{CBS}}{\geq} \\ &\frac{(\sum_{\text{cyc}} x)^2}{\sqrt{2(\sum_{\text{cyc}} x)(\sum_{\text{cyc}} xy)}} \geq \frac{(\sum_{\text{cyc}} x)^2}{\sqrt{2(\sum_{\text{cyc}} x) \cdot \frac{(\sum_{\text{cyc}} x)^2}{3}}} = \frac{9}{\sqrt{6 \cdot \frac{9}{3}}} \\ &\left(\because \sum_{\text{cyc}} x = 3r, \sum_{\text{cyc}} \frac{1}{r_a} = \frac{3r}{r} = 3 \right) = \frac{3}{\sqrt{2}} \Rightarrow (*) \text{ is true and so,} \end{aligned}$$

$$\sum_{\text{cyc}} \frac{1}{r_a \cdot \sqrt{\frac{1}{r_b} + \frac{1}{r_c}}} \geq \sqrt{\frac{3}{2r}} \quad \forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$