ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds:

$$\sum_{cyc}\frac{1}{h_a.\sqrt{\frac{1}{h_b}+\frac{1}{h_c}}}\ge\sqrt{\frac{3}{2r}}$$

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$$\sum_{cyc}\frac{1}{h_a.\sqrt{\frac{1}{h_b}+\frac{1}{h_c}}}\overset{?}{\ge}\sqrt{\frac{3}{2r}}\Leftrightarrow\sum_{cyc}\frac{1}{\frac{3r}{x}.\sqrt{\frac{y+z}{3r}}}\overset{?}{\ge}\sqrt{\frac{3}{2r}}\left(x=\frac{3r}{h_a},y=\frac{3r}{h_b},z=\frac{3r}{h_c}\right)$$

$$\Leftrightarrow\sum_{cyc}\frac{x}{\sqrt{y+z}}\underbrace{\overset{?}{\ge}}_{(*)}\frac{3}{\sqrt{2}}$$

Indeed, $$\sum_{cyc}\frac{x}{\sqrt{y+z}}=\sum_{cyc}\frac{x^2}{x.\sqrt{y+z}}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}\left(\sqrt{x}.\sqrt{xy+zx}\right)}\overset{\text{CBS}}{\ge}$$

$$\frac{\left(\sum_{cyc}x\right)^2}{\sqrt{2\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)}}\ge\frac{\left(\sum_{cyc}x\right)^2}{\sqrt{2\left(\sum_{cyc}x\right).\frac{\left(\sum_{cyc}x\right)^2}{3}}}=\frac{9}{\sqrt{6.\frac{9}{3}}}$$

$$\left(\because\sum_{cyc}x=3r.\sum_{cyc}\frac{1}{h_a}=\frac{3r}{r}=3\right)=\frac{3}{\sqrt{2}}\Rightarrow(*)\text{ is true } and \text{ so,}$$

$$\sum_{cyc}\frac{1}{h_a.\sqrt{\frac{1}{h_b}+\frac{1}{h_c}}}\ge\sqrt{\frac{3}{2r}}\ \forall\,\Delta\,\text{ABC},''=''\text{ iff }\Delta\,\text{ABC is equilateral (QED)}$$