ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$1+\sum_{cyc}\frac{r^2}{{h_a}^2}\ge 4\sum_{cyc}\frac{r^2}{h_a h_b}$$

Proposed by Marin Chirciu-Romania

Solution by Jenish Rijal-Nepal

$$\frac{r}{h_a}+\frac{r}{h_b}+\frac{r}{h_c}=1\quad\left[\because\frac{r}{h_a}+\frac{r}{h_b}+\frac{r}{h_c}=r\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)=r\cdot\frac{1}{r}=1\right]$$

$$\Rightarrow\left(\frac{r}{h_a}+\frac{r}{h_b}+\frac{r}{h_c}\right)^2=1^2=1\Rightarrow\sum_{cyc}\frac{r^2}{{h_a}^2}+2\cdot\sum_{cyc}\frac{r^2}{h_a h_b}=1$$

$$\Rightarrow 4\sum_{cyc}\frac{r^2}{h_a h_b}=2-2\sum_{cyc}\frac{r^2}{{h_a}^2}$$

Substituting this into our original inequality:

$$1+\sum_{cyc}\frac{r^2}{{h_a}^2}\ge 2-2\sum_{cyc}\frac{r^2}{{h_a}^2}\Leftrightarrow 3\sum_{cyc}\frac{r^2}{{h_a}^2}\ge 1\Leftrightarrow\sum_{cyc}\frac{r^2}{{h_a}^2}\ge\frac{1}{3}$$

Thus it suffices to show that: $\sum_{cyc}\frac{r^2}{{h_a}^2}\ge\frac{1}{3}$

$$\sum_{cyc}\frac{r^2}{{h_a}^2}=\sum_{cyc}\frac{\left(\frac{r}{h_a}\right)^2}{1}\overset{BERGSTROM}{\overbrace{\ge}}\frac{\left(\sum_{cyc}\frac{r}{h_a}\right)^2}{3}=\frac{1^2}{3}=\frac{1}{3}$$

$$\therefore 1+\sum_{cyc}\frac{r^2}{{h_a}^2}\ge 4\sum_{cyc}\frac{r^2}{h_a h_b}$$

Equality holds if and only if the triangle is equilateral.