ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{1}{3r} \le \sum_{cyc} \frac{\tan^2\frac{A}{2}}{h_a} \le \frac{8R^2-23r^2}{27r^3}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{\tan^2\frac{A}{2}}{h_a} = 2R.\sum_{cyc} \frac{(s-b)(s-c)}{s(s-a)bc} = 2R.\sum_{cyc} \frac{-s(s-a)+bc}{s(s-a)bc}$$

$$= 2R\left(\frac{-2s}{4Rrs} + \frac{1}{r^2s^2}\sum_{cyc}(s-b)(s-c)\right) = 2R\left(-\frac{1}{2Rr} + \frac{4R+r}{rs^2}\right) = \frac{8R^2+2Rr-s^2}{rs^2}$$

$$\overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Mitrinovic}}}{\le} \frac{8R^2+2Rr-16Rr+5r^2}{27r^3} \overset{\text{Euler}}{\le} \frac{8R^2-28r^2+5r^2}{27r^3} \text{ and so,}$$

$$\sum_{cyc} \frac{\tan^2\frac{A}{2}}{h_a} \le \frac{8R^2-23r^2}{27r^3} \text{ and again, } \sum_{cyc} \frac{\tan^2\frac{A}{2}}{h_a} = \frac{8R^2+2Rr-s^2}{rs^2} \overset{?}{\ge} \frac{1}{3r}$$

$$\Leftrightarrow 2s^2 \overset{?}{\le} 12R^2+3Rr \text{ and indeed, } 2s^2 \overset{\text{Gerretsen}}{\le} 8R^2+8Rr+6r^2 \overset{?}{\le} 12R^2+3Rr$$

$$\Leftrightarrow (4R+3r)(R-2r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r \text{ and so, } \sum_{cyc} \frac{\tan^2\frac{A}{2}}{h_a} \ge \frac{1}{3r}$$

$$\therefore \frac{1}{3r} \le \sum_{cyc} \frac{\tan^2\frac{A}{2}}{h_a} \le \frac{8R^2-23r^2}{27r^3}\ \forall \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$