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In any Δ ABC the following relationship holds :

$$54r\left(1-\frac{r}{R}\right)\le\sum_{cyc}h_a\cot^2\frac{A}{2}\le\frac{27R}{2}\left(\frac{R}{r}-1\right)^2$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}h_a\cot^2\frac{A}{2}=\sum_{cyc}\left(\frac{bc}{2R}\cdot\frac{s(s-a)^2}{(s-b)(s-c)(s-a)}\right)$$

$$=\frac{1}{2Rr^2}\cdot\sum_{cyc}\left(bc(s^2-2sa+a^2)\right)=\frac{s^2(s^2+4Rr+r^2)-4s.4Rrs}{2Rr^2}$$

$$=\frac{s^2(s^2-12Rr+r^2)}{2Rr^2}\overset{\substack{\text{Mitrinovic}\\ \text{and}\\ \text{Gerretsen}}}{\le}\frac{\frac{27R^2}{4}(4R^2-8Rr+4r^2)}{2Rr^2}=\frac{27R(R-r)^2}{2r^2}$$

$$=\frac{27R}{2}\left(\frac{R}{r}-1\right)^2\therefore\sum_{cyc}h_a\cot^2\frac{A}{2}\le\frac{27R}{2}\left(\frac{R}{r}-1\right)^2\ \text{and again,}$$

$$\sum_{cyc}h_a\cot^2\frac{A}{2}=\frac{s^2(s^2-12Rr+r^2)}{2Rr^2}\overset{\substack{\text{Mitrinovic}\\ \text{and}\\ \text{Gerretsen}}}{\ge}\frac{27r^2(4Rr-4r^2)}{2Rr^2}=54r\left(1-\frac{r}{R}\right)$$

$$\therefore\sum_{cyc}h_a\cot^2\frac{A}{2}\ge54r\left(1-\frac{r}{R}\right)\ \text{and so,}\ 54r\left(1-\frac{r}{R}\right)\le\sum_{cyc}h_a\cot^2\frac{A}{2}\le\frac{27R}{2}\left(\frac{R}{r}-1\right)^2$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)