ROMANIAN MATHEMATICAL MAGAZINE

In any acute $\Delta$ ABC the following relationship holds :

$$3(p^2-(2R+r)^2)\le\sum_{cyc}\frac{AH.BH^2}{AH+BH}\le 6(R-r)^2$$

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$$\Delta\ ABC\text{ is acute}\Rightarrow AH=2R\cos A>0\text{ and analogs}$$

$$\therefore\sum_{cyc}\frac{AH.BH^2}{AH+BH}\overset{AM-GM}{\ge}3.\sqrt[3]{\frac{\prod_{cyc}AH^3}{\prod_{cyc}(AH+BH)}}\overset{AM-GM}{\ge}3\left(\prod_{cyc}AH\right).\frac{3}{\sum_{cyc}(AH+BH)}$$

$$=9.8R^3.\frac{p^2-(2R+r)^2}{4R^2}.\frac{1}{2R}.\frac{1}{2\sum_{cyc}\cos A}\ge 9.8R^3.\frac{p^2-(2R+r)^2}{4R^2}.\frac{1}{2R}.\frac{1}{2.\frac{3}{2}}$$

$$\therefore\sum_{cyc}\frac{AH.BH^2}{AH+BH}\ge 3(p^2-(2R+r)^2)\text{ and again,}$$

$$\sum_{cyc}\frac{AH.BH^2}{AH+BH}\overset{AM-GM}{\le}\sum_{cyc}\frac{(AH+BH)^2.BH}{4(AH+BH)}=\frac{1}{8}\left(2\sum_{cyc}AH^2+\left(\sum_{cyc}AH\right)^2-\sum_{cyc}AH^2\right)$$

$$=\frac{1}{8}\left(4R^2\left(3-\sum_{cyc}\sin^2A\right)+4R^2.\frac{(R+r)^2}{R^2}\right)$$

$$=\frac{1}{8}\left(12R^2+4(R+r)^2-2(p^2-4Rr-r^2)\right)\overset{Gerretsen}{\le}$$

$$\frac{1}{8}\left(12R^2+4(R+r)^2-2(16Rr-5r^2-4Rr-r^2)\right)=2(R^2-Rr+r^2)\overset{?}{\le}6(R-r)^2$$

$$\Leftrightarrow 2R^2-5Rr+2r^2\overset{?}{\ge}0\Leftrightarrow(2R-r)(R-2r)\overset{?}{\ge}0\to\text{true}\because R\overset{Euler}{\ge}2r$$

$$\therefore\sum_{cyc}\frac{AH.BH^2}{AH+BH}\le 6(R-r)^2\text{ and so, }3(p^2-(2R+r)^2)\le\sum_{cyc}\frac{AH.BH^2}{AH+BH}\le 6(R-r)^2$$

$$\forall\text{ acute }\Delta\ ABC,''=''\text{ iff }\Delta\ ABC\text{ is equilateral (QED)}$$