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In any Δ ABC the following relationship holds :

$$108r^2(R-r) \le \sum_{cyc} a^2 r_a \le 27R^2(R-r)$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} a^2 r_a = \sum_{cyc}\left(a^2 . \frac{rs}{s-a}\right) = \frac{rs}{r^2 s} . \sum_{cyc}\left(a^2(-s^2+sa+bc)\right)$$

$$= \frac{-2s^2(s^2-4Rr-r^2)+2s^2(s^2-6Rr-3r^2)+8Rrs^2}{r} = \frac{4s^2(R-r)}{r} \text{ and } \because$$

$$108r^2 \overset{\text{Mitrinovic}}{\le} 4s^2 \overset{\text{Mitrinovic}}{\le} 27R^2 \therefore 108r^2(R-r) \le \sum_{cyc} a^2 r_a \le 27R^2(R-r)$$

$\forall\, \Delta$ ABC, $''='' $ iff Δ ABC is equilateral (QED)